

Estimation of Lifetime Parameters for Generalized Gompertz Distribution Under Progressively First-Failure-Censored Data

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ABSTRACT

Maximum likelihood and Bayes estimators are obtained for the three-parameter generalized Gompertz distribution, as well as the survival and hazard rate functions, using progressive first-failure censoring scheme. Two-sided approximate confidence intervals for the unknown parameters and reliability characteristics are constructed. Using gamma conjugate priors, the Bayes estimators are developed relative to both symmetric and asymmetric loss functions. However, since the Bayes estimators cannot be obtained in explicit forms, Markov chain Monte Carlo techniques are used in order to compute the Bayes estimates and associated highest posterior density credible intervals. A practical example using real-life data set is discussed for illustration purposes. Finally, to assess the performance of the proposed estimators, numerical results using Monte Carlo simulation study were reported.

Keywords: Bayesian estimator, generalized Gompertz distribution, Gibbs and Metropolis-Hastings sampler, maximum likelihood estimator, progressively first-failure-censored data.

2000 Mathematics Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05.

1 Introduction

In life-testing experiments and reliability studies, the experimenters wish to observe the failure times of units placed on test. But due to time or cost or due to some other unavoidable circumstances, experimenters terminate a study before recording the responses of all the units

under study. Data obtained from such experiments are called censored data. Type-I (time censoring) and Type-II (failure censoring) are the most common and popular censoring schemes. (Johnson, 1964) proposed a life-test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until the first-failure in each set. If the lifetime of an item is very high and test facilities are limited but test material is relatively less expensive, (Balasooriya, 1995) developed the failure-censored sampling plan by testing $k \times N$ units as N groups, each containing k units. However, the drawback of the first-failure censoring is that it cannot be to allow for sets to be removed from the test at the points other than the final termination point. Intermediate removal of survival sets during the life-test may be desirable when a compromise between the reduced time of experimentation and the observation of at least some extreme lifetimes is sought or when some of the surviving units in the experiment that are removed early on can be used for some other tests, for more detail, see (Balakrishnan and Cramer, 2014). For this reason, (Wu and Kuş, 2009) proposed a life-test plan called progressively first-failure censored sampling (PFFCS) by allowing to remove some of survival sets during the life-test. This censoring plan is a combination between the concepts of first-failure censoring and Type-II PCS. In other words, an extend of Type-II PCS is referred to as PFFCS.

The main aim of this article is to focus on the designing problem of a progressive first-failure censoring life test with a generalized Gompertz lifetime distribution. Furthermore, an extension of some literature works has been done and many special cases from the new results are obtained. The rest of this article is organized as follows: In Section 2, the generalized Gompertz model and formulation of a PFFCS, as introduced by (Wu and Kuş, 2009), are described. The maximum likelihood and Bayesian estimates of the parameters, survival and hazard functions of generalized Gompertz distribution (GGD) based on PFFCS, are investigated in Sections 3 and 4, respectively. As expected, the Bayes estimators (BEs) cannot be obtained in closed form, therefore, we propose to use the MCMC approach with the Gibbs and Metropolis-Hastings (M-H) sampling procedures to carried out the BEs and the highest posterior density (HPD) credible intervals of the unknown parameters or any function of them. Section 5 deals with a real-life data set for illustration purposes. A Monte Carlo simulation results are presented in Section 6. Finally, in Section 7, we provide some concluding remarks.

2 The GGD and PFFCS

Gompertz distribution (GD) was originally introduced by (Gompertz, 1825), is one of classical mathematical models that represent survival function based on laws of mortality. (El-Gohary, Alshamrani and Al-Otaibi, 2013) proposed a new lifetime model called the GGD, with the bathtub shape or increasing failure rate function, as an extension of the GD to provide a good fit for the real data than other well-known distributions such as: exponential distribution (ED), GD and generalized-exponential distribution (GED). The lifetime random variable $X \sim \text{GGD}(\alpha, \beta, \lambda)$ has a three-parameter GGD with scale parameter λ and shape parameters α and β , if its probability density function (PDF) $f(\cdot)$, cumulative distribution function (CDF) $F(\cdot)$, survival function

(SF) $S(\cdot)$ and hazard rate function (HRF) $H(\cdot)$ are given, respectively, by

$$f(x; \alpha, \beta, \lambda) = \alpha \lambda \exp \left(\beta x - \frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \left[1 - \exp \left(-\frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \right]^{\alpha-1},$$

$$x \geq 0, \beta \geq 0, \alpha, \lambda > 0, \quad (2.1)$$

$$F(x; \alpha, \beta, \lambda) = \left[1 - \exp \left(-\frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \right]^{\alpha},$$

$$x \geq 0, \beta \geq 0, \alpha, \lambda > 0, \quad (2.2)$$

$$S(x; \alpha, \beta, \lambda) = 1 - \left[1 - \exp \left(-\frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \right]^{\alpha},$$

$$x \geq 0, \beta \geq 0, \alpha, \lambda > 0, \quad (2.3)$$

$$\text{and } H(x; \alpha, \beta, \lambda) = \frac{\alpha \lambda \exp \left(\beta x - \frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \left[1 - \exp \left(-\frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \right]^{\alpha-1}}{1 - \left[1 - \exp \left(-\frac{\lambda}{\beta} (e^{\beta x} - 1) \right) \right]^{\alpha}},$$

$$x \geq 0, \beta \geq 0, \alpha, \lambda > 0. \quad (2.4)$$

The GGD contains some well-known distributions can be obtained as a special cases, if putting $\beta = 0$, the two-parameter GED(α, λ) can be derived; if putting $\alpha = 1$, the two-parameter GD(λ, β) can be derived; and the one-parameter ED(λ) can be derived by setting $\beta = 0$ and $\alpha = 1$. Further, the HRF of GGD (2.4) is a decreasing function with $\beta = 0$ and $\alpha < 1$, it is increasing function for $\alpha > 1$ and it is bathtub for $\beta > 0$ and $\alpha < 1$, for more detail, see (El-Gohary et al., 2013). Recently, the GGD was discussed by many several authors. (Abu-Zinadah, 2014) derived the BEs of the shape parameter of GGD under complete and Type-II censoring, (Abu-Zinadah and Bakoban, 2017) obtained the maximum likelihood estimators (MLEs) and BEs for two shape parameters and reliability characteristic of GGD based on Type-II PCS, (Ahmed, 2015) studied the maximum likelihood and Bayesian estimations of parameters of a GGD based on Type-II PCS, also, (Demir and Saraçoğlu, 2015) obtained the MLEs of the unknown parameters, and reliability characteristics of the GGD based on PFFCS. Also, they fitted the GGD to the complete real-life data set and compare its fitting with ED, GD, and GED. Furthermore, based on PFFCS, (Mohammed, Ateya and AL-Hussaini, 2017) and (Soliman, Abd-Allah, Abou-Elheggag and Abd-Elmougod, 2012) obtained the MLEs and BEs of the unknown parameters of the GED and GD, respectively, also, (Soliman and Al Sobhi, 2015) obtained the MCMC BEs of GD under PFFCS and compared with other different censoring schemes.

The PFFCS can be describe as follows: Suppose that a random sample of n units are put on a life-testing experiment at time zero as $n = N \times k$, where N independent groups with k items within each group, m is a pre-fixed number of failures, and the PCS, $\mathbf{R} = (R_1, R_2, \dots, R_m)$, is pre-fixed. At the time of first-failure observed (say $X_{(1)}$), R_1 groups and the group in which the first-failure are randomly removed from the remaining live groups $N - R_1 - 1$. Following, at the time of second-failure observed (say $X_{(2)}$), R_2 groups and the group in which the second-failure are randomly removed from the remaining live groups $N - R_1 - R_2 - 2$, and so on. This procedure continues until all remaining live R_m groups and the group in which the m -th failure (say $X_{(m)}$) has occurred are removed at the time of the observed failure m -th.

Then $X_{(1)}, X_{(2)}, \dots, X_{(m)}$ represents the independent lifetimes PFFCS order statistics with prefixed progressive censoring $\mathbf{R}$. Recently, this censoring scheme have become quite popular in reliability and lifetime testing studies. If the failure times of the $N \times k$ items originally in the life-test are from a continuous population, then the likelihood function for $X_{(i)}$, $i = 1, 2, \dots, m$, is given by

$$L(\theta|data) = Ck^m \prod_{i=1}^m f(x_{(i)}; \theta) [1 - F(x_{(i)}; \theta)]^{k(R_i+1)-1}, \quad (2.5)$$

where $C = N(N - R_1 - 1)(N - R_1 - R_2 - 2) \dots (N - \sum_{i=1}^m (R_i + 1))$. From (2.5), some sampling schemes can be obtained as a special cases, such as: Type-II PCS by putting $k = 1$, the joint PDF of the first-failure-censored order statistics by putting $\mathbf{R} = (0, 0, \dots, 0)$, Type-II censoring by putting $\mathbf{R} = (0, 0, \dots, n - m)$ and $k = 1$, if putting $\mathbf{R} = (0, 0, \dots, 0)$ and $k = 1$, then $n = m$, which yields the complete sampling. Notice that $N = n$ at $k = 1$.

3 Maximum Likelihood Estimators

Suppose that $N \times k$ independent units taken from a population are placed on a PFFCS life-test with the corresponding lifetimes being identically distributed having PDF and CDF of GGD as defined in (2.1) and (2.2), respectively. Substituting (2.1) and (2.2) into (2.5), the proportional likelihood function of (2.5) can be written as

$$L(\varpi|data) \propto (\alpha\lambda)^m \exp\left(\beta m\bar{x} - \lambda \sum_{i=1}^m \varphi(x_{(i)}; \beta)\right) \prod_{i=1}^m [1 - \exp(-\lambda \varphi(x_{(i)}; \beta))]^{\alpha-1} \\ \times [1 - [1 - \exp(-\lambda \varphi(x_{(i)}; \beta))]^{\alpha}]^{k(R_i+1)-1}, \quad (3.1)$$

where $\varpi = (\alpha, \beta, \lambda)^T$ is the parameter vector and $\varphi(x_{(i)}; \beta) = \frac{1}{\beta} (\exp(\beta x_{(i)}) - 1)$, $i = 1, 2, \dots, m$.

Using the general binomial expansion series $(1 - (1 - x)^y)^z = \sum_{d=0}^z (-1)^d \binom{z}{d} (1 - x)^{yd}$, hence, equation (3.1) is equivalent to

$$L(\varpi|data) \propto (\alpha\lambda)^m \exp\left(\beta m\bar{x} - \lambda \sum_{i=1}^m \varphi(x_{(i)}; \beta)\right) \\ \times \prod_{i=1}^m \sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda \varphi(x_{(i)}; \beta))]^{\alpha(d+1)-1}, \quad (3.2)$$

where $\bar{x} = \sum_{i=1}^m x_{(i)}/m$ and $Q_d = (-1)^d \binom{k(R_i+1)-1}{d}$, $i = 1, 2, \dots, m$.

The corresponding log-likelihood function of (3.2), $\ell(\cdot) = \log L(\cdot)$, becomes

$$\ell(\varpi|data) \propto m \log(\alpha\lambda) + \beta m\bar{x} - \lambda \sum_{i=1}^m \varphi(x_{(i)}; \beta) \\ + \sum_{i=1}^m \log \left(\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda \varphi(x_{(i)}; \beta))]^{\alpha(d+1)-1} \right). \quad (3.3)$$

Differentiating (3.3) with respect to α , β and λ , respectively, we have

$$\frac{\partial \ell}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^m \left\{ \frac{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]^{\alpha(d+1)-1} \log [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]}{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]^{\alpha(d+1)-1}} \right\},$$

$$\frac{\partial \ell}{\partial \beta} = m\bar{x} - \lambda \sum_{i=1}^m \varphi'(x_{(i)}; \beta) + \sum_{i=1}^m \left\{ \frac{\sum_{d=0}^{k(R_i+1)-1} Q_d (\alpha(d+1) - 1) \varphi'(x_{(i)}; \beta) e^{-\lambda \varphi(x_{(i)}; \beta)} [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]^{\alpha(d+1)-2}}{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]^{\alpha(d+1)-1}} \right\},$$

and

$$\frac{\partial \ell}{\partial \lambda} = \frac{m}{\lambda} - \sum_{i=1}^m \varphi(x_{(i)}; \beta) + \sum_{i=1}^m \left\{ \frac{\sum_{d=0}^{k(R_i+1)-1} Q_d (\alpha(d+1) - 1) \varphi(x_{(i)}; \beta) e^{-\lambda \varphi(x_{(i)}; \beta)} [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]^{\alpha(d+1)-2}}{\sum_{d=0}^{k(R_i+1)-1} Q_d [1 - e^{-\lambda \varphi(x_{(i)}; \beta)}]^{\alpha(d+1)-1}} \right\}, \quad (3.4)$$

where $\varphi'(x_{(i)}; \beta) = -\frac{1}{\beta} [\varphi(x_{(i)}; \beta) - x_{(i)} \exp(\beta x_{(i)})]$, $i = 1, 2, \dots, m$. Equating each result in (3.4) to zero, three equations must be simultaneously satisfied to obtain the MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ of the three unknown parameters of GGD α , β and λ , respectively. Clearly, the MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ cannot be solved analytically and can be obtained by solving set of nonlinear equations, therefore, it can be used numerical methods such as Newton-Raphson to solve these equations. Furthermore, once the estimates of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ obtained, using invariance property of MLE, then the MLEs $\hat{S}(t)$ and $\hat{H}(t)$ of $S(t)$ and $H(t)$ as in (2.3) and (2.4), respectively, for given mission time t , can be derived by replacing α , β and λ by their MLEs $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$, respectively.

3.1 Approximate interval estimation

The two-sided approximate confidence intervals (ACIs) for the unknown parameters α , β and λ based on the asymptotic normal approximation of the MLEs as well as for SF, $S(t)$, and HRF, $H(t)$, of GGD based on the delta method are constructed. From (3.4), the Fisher information matrix, $I_0^{-1}(\underline{\theta})$, is given by

$$I_{ij}(\underline{\theta}) = E[-(\partial^2 \ln L(\underline{\theta} | \underline{x}) / \partial \theta_i \partial \theta_j)], \quad i, j = 1, 2, 3. \quad (3.5)$$

Since, the exact mathematical expressions for the expectation of the Fisher information matrix (3.5) are very difficult to obtain. Hence, the Fisher information matrix can be approximated by

$$I_{ij}(\underline{\theta}) \cong [-(\partial^2 \ln L(\underline{\theta} | \underline{x}) / \partial \theta_i \partial \theta_j)]_{(\varpi = \hat{\varpi})}, \quad i, j = 1, 2, 3.$$

which is obtained by dropping E and replaced $\varpi = (\alpha, \beta, \lambda)$ with $\hat{\varpi} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$, respectively, for more detail, see (Cohen, 1965). The approximate variance-covariance (V-Cov) matrix is obtained by inverting the Fisher information matrix, practically, estimating $I_0^{-1}(\varpi)$ by $I_0^{-1}(\hat{\varpi})$, as

$$I_0^{-1}(\hat{\varpi}) \cong \begin{bmatrix} -\frac{\partial^2 \ell}{\partial \alpha^2} & -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ell}{\partial \beta^2} & -\frac{\partial^2 \ell}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ell}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ell}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ell}{\partial \lambda^2} \end{bmatrix}_{(\varpi = \hat{\varpi})}^{-1} = \begin{bmatrix} \hat{\sigma}_{\hat{\alpha}}^2 & \hat{\sigma}_{\hat{\alpha}, \hat{\beta}} & \hat{\sigma}_{\hat{\alpha}, \hat{\lambda}} \\ \hat{\sigma}_{\hat{\beta}, \hat{\alpha}} & \hat{\sigma}_{\hat{\beta}}^2 & \hat{\sigma}_{\hat{\beta}, \hat{\lambda}} \\ \hat{\sigma}_{\hat{\lambda}, \hat{\alpha}} & \hat{\sigma}_{\hat{\lambda}, \hat{\beta}} & \hat{\sigma}_{\hat{\lambda}}^2 \end{bmatrix}. \quad (3.6)$$

Clearly, the second partial derivatives from the natural logarithm likelihood function (8) has a nonlinear equations, hence, the approximate V-Cov matrix (3.3) can be obtained numerically. Under regularity conditions for the asymptotic properties of the MLEs of the three-parameter GGD, the asymptotic normality of the MLEs $\hat{\omega}$ is approximately multivariate normal with mean ω and approximate asymptotic V-Cov matrix $I_0^{-1}(\hat{\omega})$, i.e., $\hat{\omega} \sim (\omega, I_0^{-1}(\hat{\omega}))$, see (Lawless, 2003). Hence, the $100(1 - \gamma)\%$ two-sided ACIs for the three-parameter GGD α , β and λ , based on PFFCS are given, respectively, by

$$\left(\hat{\alpha} \pm z_{\gamma/2} \sqrt{\hat{\sigma}_{\hat{\alpha}}^2} \right), \quad \left(\hat{\beta} \pm z_{\gamma/2} \sqrt{\hat{\sigma}_{\hat{\beta}}^2} \right) \quad \text{and} \quad \left(\hat{\lambda} \pm z_{\gamma/2} \sqrt{\hat{\sigma}_{\hat{\lambda}}^2} \right),$$

where $\hat{\sigma}_{\hat{\alpha}}^2$, $\hat{\sigma}_{\hat{\beta}}^2$ and $\hat{\sigma}_{\hat{\lambda}}^2$, are the elements on the main diagonal of the asymptotic variance-covariance matrix (3.6), respectively, and $z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $(\gamma/2) - th$.

Furthermore, to construct the asymptotic $100(1 - \gamma)\%$ two-sided ACIs of the SF and HRF of GGD based on PFFCS, which they are functions in the parameters α , β and λ , we need to find the variances of them. Therefore, the delta method is considered to obtain the approximate estimates of the variance of $\hat{S}(t)$ and $\hat{H}(t)$. The delta method is a general approach for computing ACIs for functions of MLEs, for more detail, see (Greene, 2012). According to delta method, the variance $\hat{\sigma}_{\hat{S}(t)}^2$ and $\hat{\sigma}_{\hat{H}(t)}^2$ of $\hat{S}(t)$ and $\hat{H}(t)$ can be approximated, respectively by

$$\hat{\sigma}_{\hat{S}(t)}^2 = [\nabla \hat{S}(t)]^T I_0^{-1}(\hat{\omega}) [\nabla \hat{S}(t)] \quad \text{and} \quad \hat{\sigma}_{\hat{H}(t)}^2 = [\nabla \hat{H}(t)]^T I_0^{-1}(\hat{\omega}) [\nabla \hat{H}(t)],$$

where

$$[\nabla \hat{S}(t)]^T = [\partial \nabla S(t) / \partial \alpha, \partial \nabla S(t) / \partial \beta, \partial \nabla S(t) / \partial \lambda]_{(\alpha=\hat{\alpha}, \beta=\hat{\beta}, \lambda=\hat{\lambda})},$$

and

$[\nabla \hat{H}(t)]^T = [\partial \nabla H(t) / \partial \alpha, \partial \nabla H(t) / \partial \beta, \partial \nabla H(t) / \partial \lambda]_{(\alpha=\hat{\alpha}, \beta=\hat{\beta}, \lambda=\hat{\lambda})}$, are the gradient of $\hat{S}(t)$ and $\hat{H}(t)$, respectively, with respect to α , β and λ . Hence, the $100(1 - \gamma)\%$ two-sided ACIs of $S(t)$ and $H(t)$, are given, respectively, by

$$\left(\hat{S}(t) \pm z_{\gamma/2} \sqrt{\hat{\sigma}_{\hat{S}(t)}^2} \right) \quad \text{and} \quad \left(\hat{H}(t) \pm z_{\gamma/2} \sqrt{\hat{\sigma}_{\hat{H}(t)}^2} \right),$$

where $z_{\gamma/2}$ is the percentile of standard normal distribution with right-tail probability $(\gamma/2) - th$.

4 Bayes Estimators

In this section, the Bayes estimators $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\lambda}$ can be developed for the generalized Gompertz parameters α , β and λ , as well as, SF, $S(t)$, and, HRF, $H(t)$, against squared error loss (SEL), linear-exponential loss (LL) and general entropy loss (GEL) functions based on PFFCS. The most commonly symmetric loss function is the SEL function (which is denoted by $\ell_S(\phi, \tilde{\phi})$), see (Martz and Waller, 1982), is defined as

$$l_S(\phi, \tilde{\phi}) = (\tilde{\phi} - \phi)^2. \quad (4.1)$$

Under SEL function (4.1), the BE $\tilde{\phi}_S(\alpha, \beta, \lambda)$ (say) of any function of unknown parameters $\phi(\alpha, \beta, \lambda)$ (say), is the posterior mean, it is given by

$$\tilde{\phi}_S = E[\phi(\alpha, \beta, \lambda)|\underline{x}].$$

The LL function (which is denoted by $\tilde{\phi}_L(\alpha, \beta, \lambda)$) and GEL function (which is denoted by $\tilde{\phi}_G(\alpha, \beta, \lambda)$) are the most commonly asymmetric loss functions. For LL was proposed by (Varian, 1975), and it is given by

$$l_L(\phi, \tilde{\phi}) = e^{h\Delta} - h\Delta - 1, \quad h \neq 0, \quad (4.2)$$

where $\Delta = (\tilde{\phi} - \phi)$. The sign, and magnitude of the shape parameter h represent the direction and degree of symmetry respectively (if $h > 0$, overestimation is more serious than underestimation and $h < 0$ means the opposite). For h close to zero, the LL function is approximately the SEL function and therefore almost symmetric. Under (4.2), the BE $\tilde{\phi}$ of ϕ is given by

$$\tilde{\phi}_L = -\frac{1}{h} \log \left[E_{\phi} \left(e^{-h\phi} | \underline{x} \right) \right],$$

provided the above exception exists, and is finite, for more detail see (Zellner, 1986).

Another useful asymmetric loss function is the GEL function, is defined as

$$l_G(\phi, \tilde{\phi}) \propto \left(\frac{\tilde{\phi}}{\phi} \right)^q - q \log \left(\frac{\tilde{\phi}}{\phi} \right) - 1, \quad q \neq 0, \quad (4.3)$$

where q is a shape parameter. The minimum of errors under this loss function occurs at $\tilde{\phi} = \phi$. This loss function is a generalization of the entropy-loss used in several papers where $q = 1$ by (Dey, Ghosh and Srinivasan, 1987) and (Dey and Liu, 1992). For $q > 0$, a positive error has a more serious effect than a negative error, and for $q < 0$, a negative error has a more serious effect than a positive error. Note that for $q = -1$, the BE coincides with the BE under SEL function. Under (4.3), the BE $\tilde{\phi}$ of ϕ is given by

$$\tilde{\phi}_G = [E_{\phi} (\phi^{-q} | \underline{x})]^{-1/q},$$

provided the above exception exists, and it is finite.

According to (Ahmed, 2015), we consider independent informative priors of α , β and λ with gamma distributions, respectively, as

$$\begin{aligned} \pi(\alpha) &\propto \alpha^{a_1-1} e^{-b_1\alpha}, & \alpha > 0, \quad a_1, b_1 > 0, \\ \pi(\beta) &\propto \beta^{a_2-1} e^{-b_2\beta}, & \beta > 0, \quad a_2, b_2 > 0, \\ \text{and} \quad \pi(\lambda) &\propto \lambda^{a_3-1} e^{-b_3\lambda}, & \lambda > 0, \quad a_3, b_3 > 0, \end{aligned} \quad (4.4)$$

where all hyper-parameters a_i and b_i , $i = 1, 2, 3$, are assumed to be non-negative and known. From (4.4), the joint prior density of α , β and λ becomes

$$\pi(\alpha, \beta, \lambda) \propto \alpha^{a_1-1} \beta^{a_2-1} \lambda^{a_3-1} e^{-(b_1\alpha+b_2\beta+b_3\lambda)}, \quad \alpha, \beta, \lambda > 0, \quad a_1, a_2, a_3, b_1, b_2, b_3 > 0. \quad (4.5)$$

Combining (3.2) and (4.5), the posterior distribution of the parameters α , β and λ can be written up to proportional as

$$\begin{aligned} \pi(\varpi|\underline{x}) &\propto \alpha^{a_1+m-1} \beta^{a_2-1} \lambda^{a_3+m-1} \\ &\times \exp\left(-b_1\alpha - \beta(b_2 - m\bar{x}) - \lambda\left(b_3 + \sum_{i=1}^m \varphi(x_i; \beta)\right)\right) \\ &\times \prod_{i=1}^m \sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda\varphi(x_i; \beta))]^{\alpha(d+1)-1}. \end{aligned} \quad (4.6)$$

The normalizing constant C of (4.6) is given by

$$\begin{aligned} C &= \int_0^\infty \int_0^\infty \int_0^\infty \alpha^{a_1+m-1} \beta^{a_2-1} \lambda^{a_3+m-1} \\ &\times \exp\left(-b_1\alpha - \beta(b_2 - m\bar{x}) - \lambda\left(b_3 + \sum_{i=1}^m \varphi(x_i; \beta)\right)\right) \\ &\times \prod_{i=1}^m \sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda\varphi(x_i; \beta))]^{\alpha(d+1)-1} d\alpha d\beta d\lambda. \end{aligned}$$

Generally, the ratio of multiple integrals given in (4.6) cannot be obtained in a simple closed form, this is due to the complex form of the likelihood function given in (3.2). Therefore, the BEs of α , β , λ , $S(t)$ and $H(t)$ cannot be solvable analytically but can be evaluated numerically by using computer facilities and numerical techniques. For this reason, we propose to use MCMC method to generate samples from the joint posterior density function (4.6) and use them to compute the BE of the unknown parameters and reliability characteristic of GGD, as well as, to construct the associated HPD credible intervals. To implement the MCMC methodology, we consider the Gibbs within M-H sampler procedure, which requires the derivation of the complete set of conditional posterior distribution, for more detail, see (Brooks, 1998) and (Gamerman and Lopes, 2006). From (4.6), the full conditionals for α , β and λ can be written up to proportionality, respectively, as

$$\pi_1(\alpha|\beta, \lambda; \underline{x}) \propto \alpha^{a_1+m-1} e^{-b_1\alpha} \prod_{i=1}^m \sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda\varphi(x_i; \beta))]^{\alpha(d+1)-1}, \quad (4.7)$$

$$\begin{aligned} \pi_2(\beta|\alpha, \lambda; \underline{x}) &\propto \beta^{a_2-1} \exp\left(-\beta(b_2 - m\bar{x}) - \lambda\left(b_3 + \sum_{i=1}^m \varphi(x_i; \beta)\right)\right) \\ &\times \prod_{i=1}^m \sum_{d=0}^{k(R_i+1)-1} Q_d [1 - \exp(-\lambda\varphi(x_i; \beta))]^{\alpha(d+1)-1}, \end{aligned} \quad (4.8)$$

and

$$\pi_3(\lambda|\alpha, \beta; \underline{x}) \propto \lambda^{a_3+m-1} \exp\left(-\lambda\left(b_3 + \sum_{i=1}^m \varphi(x_i; \beta)\right)\right). \quad (4.9)$$

From (4.9), the unknown parameter λ has the gamma density with shape parameter $m + a_3$ and scale parameter $b_3 + \sum_{i=1}^m \varphi(x_i; \beta)$. Thus, samples of λ can be easily generated using any gamma generating routine. Since, the conditional posterior distributions (4.7) and (4.8) of α and β , respectively, cannot be reduced analytically to well-known distributions, then the use of M-H sampler is required for the implementations of MCMC methodology.

The hybrid algorithm with Gibbs sampling steps for updating the unknown parameter λ and with M-H steps for updating the unknown parameters α and β is carried out as follows:

Step 1: Start with an initial values $\alpha^{(0)}$, $\beta^{(0)}$ and $\lambda^{(0)}$.

Step 2: Set $J = 1$.

Step 3: Generate $\lambda^{(J)}$ from $Gamma(m + a_3, b_3 + \sum_{i=1}^m \varphi(x_{(i)}; \beta))$.

Step 4: Using M-H algorithm, generate samples of $\alpha^{(J)}$ and $\beta^{(J)}$ from $\pi(\alpha^{(J-1)} | \beta^{(J-1)}, \lambda^{(J)}; \underline{x})$ and $\pi(\beta^{(J-1)} | \alpha^{(J)}, \lambda^{(J)}; \underline{x})$ with the normal proposal distributions $N(\alpha^{(J-1)}, \sigma_\alpha^2)$ and $N(\beta^{(J-1)}, \sigma_\beta^2)$, respectively, as

(a) Generate α^* from $N(\alpha^{(J-1)}, \sigma_\alpha^2)$ and β^* from $N(\beta^{(J-1)}, \sigma_\beta^2)$.

(b) Evaluate the acceptance probabilities by

$$\rho_\alpha(\alpha^{(J-1)}, \alpha^*) = \min \left[1, \frac{\pi(\alpha^* | \beta^{(J-1)}, \lambda^{(J)}; \underline{x})}{\pi(\alpha^{(J-1)} | \beta^{(J-1)}, \lambda^{(J)}; \underline{x})} \right],$$

and

$$\rho_\beta(\beta^{(J-1)}, \beta^*) = \min \left[1, \frac{\pi(\beta^* | \alpha^{(J)}, \lambda^{(J)}; \underline{x})}{\pi(\beta^{(J-1)} | \alpha^{(J)}, \lambda^{(J)}; \underline{x})} \right].$$

(c) Generate a samples u_1 and u_2 from uniform $U(0, 1)$ distribution.

(d) If $u_1 \leq \rho_\alpha$, accept the proposal and set $\alpha^{(J)} = \alpha^*$, else $\alpha^{(J)} = \alpha^{(J-1)}$.

(e) If $u_2 \leq \rho_\beta$, accept the proposal and set $\beta^{(J)} = \beta^*$, else $\beta^{(J)} = \beta^{(J-1)}$.

Step 5: Using $\alpha^{(J)}$, $\lambda^{(J)}$ and $\beta^{(J)}$ to compute the SF and HRF as in (2.3) and (2.4), for a given mission time t , respectively, as

$$S^{(J)}(t) = 1 - F(\alpha^{(J)}, \beta^{(J)}, \lambda^{(J)}; t) \text{ and } H^{(J)}(t) = \frac{f(\alpha^{(J)}, \beta^{(J)}, \lambda^{(J)}; t)}{1 - F(\alpha^{(J)}, \beta^{(J)}, \lambda^{(J)}; t)}.$$

Step 6: Set $J = J + 1$

Step 7: Repeat steps 3-5 for M times until to get M draws.

$$\phi^{(J)} = (\alpha^{(J)}, \beta^{(J)}, \lambda^{(J)}, S^{(J)}(t), H^{(J)}(t)), \quad J = 1, 2, \dots, M.$$

Step 8: To compute the HPD credible intervals of $\phi = (\alpha, \beta, \lambda, S(t), H(t))$ order the MCMC sample of $\phi^{(J)}$ for $J = 1, 2, \dots, M$, as $\alpha_{(1)}, \dots, \alpha_{(M)}$, $\beta_{(1)}, \dots, \beta_{(M)}$, $\lambda_{(1)}, \dots, \lambda_{(M)}$, $S_{(1)}(t), \dots, S_{(M)}(t)$, and $H_{(1)}(t), \dots, H_{(M)}(t)$.

Based on the method proposed by (Chen and Shao, 1999), the $100(1 - \gamma)\%$ two-sided HPD credible interval for the unknown parameters or any function of them such as SF and HRF as in ϕ , is given by $[\phi_{(J^*)}, \phi_{(J^*+(1-\gamma)M)}]$, where J^* is chosen such that

$$\phi_{(J^*+(1-\gamma)M)} - \phi_{(J^*)} = \min_{1 \leq J \leq \gamma M} (\phi_{(J+(1-\gamma)M)} - \phi_{(J)}), \quad J^* = 1, 2, \dots, M.$$

Here $[x]$ denotes the largest integer less than or equal to x . Then the HPD credible interval of ϕ is that interval which has the shortest length. The first simulated varieties, M_0 , may be biased by the initial value, therefore, usually discarded in the beginning of the analysis implementation (burn-in period) in order to guarantee the convergence and to remove the affection of the selection of initial.

Then the selected samples $\phi^{(J)}$, $J = M_0 + 1, \dots, M$, are sufficiently large M and can be used to develop the Bayesian inferences of ϕ . Hence, the BEs of ϕ based on SEL, LL and GEL functions as in (4.1), (4.2) and (4.3), respectively, are given by

$$\tilde{\phi}_S = \sum_{J=M_0+1}^M \phi^{(J)} / (M - M_0),$$

$$\tilde{\phi}_L = -\frac{1}{h} \log \left[\sum_{J=M_0+1}^M \exp \left(-h\phi^{(J)} \right) / (M - M_0) \right], \quad h \neq 0,$$

and

$$\tilde{\phi}_G = \left[\sum_{J=M_0+1}^M \left(\phi^{(J)} \right)^{-q} / (M - M_0) \right]^{-1/q}, \quad q \neq 0,$$

where M_0 is burn-in.

5 Real-Life Data Analysis

In this section, we have considered a well-known real-life data set based on 50 observed failure times from a life-test experiment, this data set was originally proposed by (Aarset, 1987), see Table 1. Also, (El-Gohary et al., 2013) observed that the GGD can be used quite effectively to analyze this data set. Recently, (Abu-Zinadah, 2014), (Abu-Zinadah and Bakoban, 2017), (Ahmed, 2015) and (Ashour, El-Sheikh and Elshahhat, 2018) reanalyzed this data set. To illustrate the inferential method developed in the preceding section, we have assumed that the failure times data of $n = 50$ devices are randomly grouped into $N = 25$ groups with $k = 2$ devices within each group, see Table 2. Finally, the following first-failure censored sample is obtained in order as: 0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 12, 18, 18, 18, 32, 36, 45, 63, 67, 67, 67, 72, 82, 83, 84, 84.

Using the first-failure censored sample, the different schemes of PFFCS and corresponding samples are reported in Table 3. For brevity, the censoring scheme $\mathbf{R} = (3, 0, 0, 0, 0, 3)$ is denoted by $\mathbf{R} = (3, 0^*4, 3)$. Because we have no prior information about the unknown parameters α , β and λ , we assume the non-informative prior. Based on the MCMC samples of size 6000 and discard 1000 as a burn-in, the BEs with their standard errors of the unknown parameters α , β and λ , as well as, reliability characteristic such as: $S(5)$ and $H(5)$, for given $t = 5$, are computed. The point estimates of MLEs and MCMC BEs are presented in Table 4. The initial values for the unknown parameters α , β and λ for running the MCMC sampler algorithm were taken to be their MLEs. Moreover, the 95% two-sided ACI/HPD credible intervals with associated lengths are computed and reported in Table 5.

From Table 4, it can be seen that, as expected the MCMC BEs and the MLE are quite close to each other. It is well known for the LL function that $h > 0$ implies that overestimation results in more penalty than underestimation and reverse is true for $h < 0$. In a similar manner it is to be noted for the GEL function that $q > 0$ means overestimation is more serious than under estimation and opposite is true for $q < 0$. It is clear that the standard errors and confidence intervals

Table 1: Failure times of 50 devices.

0.1	0.2	1	1	1	1	1	2	3	6
7	11	12	18	18	18	18	18	21	32
36	40	45	45	47	50	55	60	63	63
67	67	67	67	72	75	79	82	82	83
84	84	84	85	85	85	85	85	86	86

Table 2: Random grouping of Aarset real data set.

Group Item	1	2	3	4	5	6	7	8	9	10	11	12	13
1	21	1	45	0.2	82	85	18	67	18	45	0.1	50	82
2	6	18	55	40	36	1	47	84	12	18	1	32	85
Group Item	14	15	16	17	18	19	20	21	22	23	24	25	
1	86	2	67	86	11	67	67	18	85	1	1	85	
2	83	60	75	84	3	63	79	63	84	85	7	72	

Note: Bold observation represents to the first-failure item in a group.

lengths of the MCMC BEs for α , β , λ , $S(5)$ and $H(5)$ are smaller than their MLEs. All computations were performed using R statistical programming language software with CODA package proposed by (Plummer, Best, Cowles and Vines, 2006) and maxLik package proposed by (Henningsen and Toomet, 2011), which using the method of Newton-Raphson maximization in the maximum likelihood computations.

6 Simulation Study

In this section, a Monte Carlo simulation was conducted to examine and compare the performance of the proposed methods for different sample sizes and different censoring schemes for all parameters α , β , λ , $S(t)$ and $H(t)$ of GGD. We simulated 1000 progressively first-failure-censored samples for various true values of parameters (α, β, λ) and different combinations of N (number of groups), k (number of items within each group), m (pre-determined number of failures) from the GGD by using the algorithm described in (Balakrishnan and Sandhu, 1995).

Table 3: Three different generated samples based on PFFCS.

Censoring Scheme				Sample
N	m	k	$\mathbf{R}$	
25	15	2	$\mathbf{R}_1 = (10, 0^{*14})$	0.1, 18, 18, 32, 36, 45, 63, 67, 67, 67, 72, 82, 83, 84, 84.
			$\mathbf{R}_2 = (0^{*6}, 3, 4, 3, 0^{*6})$	0.1, 0.2, 1, 1, 1, 1, 2, 18, 45, 67, 72, 82, 83, 84, 84.
			$\mathbf{R}_3 = (0^{*14}, 10)$	0.1, 0.2, 1, 1, 1, 1, 2, 3, 6, 12, 18, 18, 18, 32, 36.

Table 4: MLEs and BEs of $\alpha, \beta, \lambda, S(5)$ and $H(5)$, with their standard errors (in parentheses).

CS	MLE	SEL	LL		GEL	
			$h = -3$	$h = 3$	$q = -3$	$q = 3$
R₁	α	0.4790 (0.1280)	0.4834 (6.2×10^{-5})	0.4833 (6.1×10^{-5})	0.4834 (6.2×10^{-5})	0.4833 (6.1×10^{-5})
	β	0.0755 (0.0110)	0.0324 (6.1×10^{-4})	0.0323 (6.1×10^{-4})	0.0327 (6.1×10^{-4})	0.0318 (6.2×10^{-4})
	λ	0.0001 (0.0001)	0.0111 (1.6×10^{-4})	0.0109 (1.5×10^{-4})	0.0158 (2.2×10^{-4})	0.0055 (7.7×10^{-5})
	$S(5)$	0.9974 (0.0042)	0.7601 (0.2247)	0.7686 (2.9×10^{-3})	0.7507 (3.1×10^{-3})	0.7677 (2.9×10^{-3})
	$H(5)$	0.0003 (0.0002)	0.0333 (0.0330)	0.0336 (4.3×10^{-4})	0.0329 (4.2×10^{-4})	0.0391 (5.0×10^{-4})
	α	0.3413 (0.0834)	0.3525 (0.1256)	0.3526 (1.8×10^{-3})	0.3524 (1.8×10^{-3})	0.3525 (1.8×10^{-3})
	β	0.0528 (0.0154)	0.0427 (0.0329)	0.0428 (4.6×10^{-4})	0.0426 (4.6×10^{-4})	0.0430 (4.6×10^{-4})
	λ	0.0002 (0.0001)	0.0034 (0.0035)	0.0035 (4.8×10^{-5})	0.0034 (4.6×10^{-5})	0.0037 (5.1×10^{-5})
	$S(5)$	0.9009 (0.0546)	0.7554 (0.2171)	0.7562 (3.0×10^{-3})	0.7545 (3.1×10^{-3})	0.7561 (3.0×10^{-3})
	$H(5)$	0.0085 (0.4520)	0.0252 (0.0220)	0.0253 (3.1×10^{-4})	0.0251 (3.1×10^{-4})	0.0255 (3.1×10^{-4})
R₃	α	0.3449 (0.0835)	0.3431 (0.1359)	0.3432 (1.9×10^{-3})	0.3431 (1.9×10^{-3})	0.3432 (1.9×10^{-3})
	β	0.0406 (0.0374)	0.0498 (0.0260)	0.0499 (3.6×10^{-4})	0.0497 (3.6×10^{-4})	0.0502 (3.6×10^{-4})
	λ	0.0005 (0.0003)	0.0183 (0.0189)	0.0184 (2.6×10^{-4})	0.0182 (2.6×10^{-4})	0.0196 (2.8×10^{-4})
	$S(5)$	0.8614 (0.0637)	0.5530 (0.4208)	0.5565 (5.9×10^{-3})	0.5498 (6.0×10^{-3})	0.5571 (5.9×10^{-3})
	$H(5)$	0.0122 (0.5380)	0.0603 (0.0578)	0.0605 (8.1×10^{-4})	0.0601 (8.0×10^{-4})	0.0620 (8.3×10^{-4})
	α	0.3449 (0.0835)	0.3431 (0.1359)	0.3432 (1.9×10^{-3})	0.3431 (1.9×10^{-3})	0.3432 (1.9×10^{-3})
	β	0.0406 (0.0374)	0.0498 (0.0260)	0.0499 (3.6×10^{-4})	0.0497 (3.6×10^{-4})	0.0502 (3.6×10^{-4})
	λ	0.0005 (0.0003)	0.0183 (0.0189)	0.0184 (2.6×10^{-4})	0.0182 (2.6×10^{-4})	0.0196 (2.8×10^{-4})
	$S(5)$	0.8614 (0.0637)	0.5530 (0.4208)	0.5565 (5.9×10^{-3})	0.5498 (6.0×10^{-3})	0.5571 (5.9×10^{-3})
	$H(5)$	0.0122 (0.5380)	0.0603 (0.0578)	0.0605 (8.1×10^{-4})	0.0601 (8.0×10^{-4})	0.0620 (8.3×10^{-4})

Table 5: The 95% two-sided ACI/HPD credible intervals of α , β , λ , $S(5)$ and $H(5)$.

Parameter		R_1		R_2		R_3	
		ACI	HPD	ACI	HPD	ACI	HPD
α	Lower	0.2286	0.4862	0.1779	0.3552	0.1814	0.3460
	Upper	0.7294	0.4801	0.5048	0.3484	0.5085	0.3377
	Length	0.5008	0.0061	0.3269	0.0068	0.3271	0.0083
β	Lower	0.0539	0.0397	0.0227	0.0479	0.0000	0.0552
	Upper	0.0970	0.0278	0.0829	0.0363	0.1139	0.0410
	Length	0.0431	0.0119	0.0602	0.0117	0.1139	0.0142
λ	Lower	0.0000	0.0300	0.0000	0.0053	0.0000	0.0244
	Upper	0.0002	0.0028	0.0004	0.0019	0.0012	0.0084
	Length	0.0002	0.0272	0.0004	0.0034	0.0012	0.0160
$S(5)$	Lower	0.9186	0.8657	0.7939	0.7934	0.7364	0.6470
	Upper	1.0000	0.5939	1.0000	0.7118	0.9864	0.4995
	Length	0.0814	0.2718	0.2061	0.0816	0.2500	0.1475
$H(5)$	Lower	0.0000	0.0668	0.0000	0.0307	0.0000	0.0722
	Upper	0.3957	0.0160	0.8954	0.0204	1.0664	0.0397
	Length	0.3957	0.0508	0.8954	0.0103	1.0664	0.0325

Arbitrary initial values of the model parameters $(\alpha, \beta, \lambda) = (1, 1, 0.1)$, are considered to compare the different BEs. Comparison between different estimators is made with respect to their mean squared error (MSE) values. Also, the performances of the two-sided 95% HPD credible intervals are compared using average lengths (ALs). The simulation study was performed for different values of (N, k, m) such as: $N = 20$ and 50 for each group size $k = 2$ and 5 . The BEs and 95% HPD credible intervals are computed for different choices of k , N and m based on 12,000 MCMC samples and discard first 2000 values as burn-in. The test is terminated when the number of failed subjects achieves or exceeds a certain value m where the failure proportion is $m/N = 30, 60$ and 90% . Using a hybrid strategy combining M-H within Gibbs algorithm described in Section 3, the different BEs are developed under three loss functions such as SEL, LL with $h = -5, 5$ and GEL with $q = -5, 5$.

In Bayesian analysis, the choice of the hyper-parameters is the main issue. Under non-informative priors $a_i = b_i = 0$ for $i = 1, 2, 3$, the joint posterior distribution of the unknown three parameters is proportional to the likelihood function. For informative priors, we have chosen the hyper-parameters in such way that the prior mean become the expected value of the corresponding unknown parameter. Hence, we have used informative priors $a_i = b_i = 1$, $i = 1, 2$, and $(a_3, b_3) = (0.1, 1)$ when $(\alpha, \beta, \lambda) = (1, 1, 0.1)$. Further, we have also obtained the MLEs and BEs of the SF and HRF where the true values of $S(t)$ and $H(t)$ for specified time $t = 0.5$, are taken as $S(0.5) = 0.9371873$ and $H(0.5) = 0.1648721$, respectively.

Now, we consider the following different censoring schemes as

$$\text{CS-I} : R_1 = N - m, \quad R_i = 0 \quad \text{for } i \neq 1,$$

$$\text{CS-II} : \begin{cases} R_{m/2} = N - m, & R_i = 0 \quad \text{for } i \neq m/2, \quad \text{if } m \text{ is even,} \\ R_{(m+1)/2} = N - m, & R_i = 0 \quad \text{for } i \neq (m+1)/2, \quad \text{if } m \text{ is odd,} \end{cases}$$

$$\text{CS-III} : R_m = N - m, \quad R_i = 0 \quad \text{for } i \neq m.$$

For each censoring scheme, average MLEs and BEs (using MCMC procedure) of $\alpha, \beta, \lambda, S(t)$ and $H(t)$ with their associated MSEs are computed and reported in Tables 6-15. Moreover, the average lengths of 95% ACI/HPD credible intervals are summarized in Tables 16-17, respectively. From the simulation results listed in Tables 6-17, it can be seen that the MLEs and BEs of unknown parameters and the reliability characteristics of GGD are very good in term of minimum MSEs. As N and m increases, the MSEs of all estimates decrease as expected. In most cases, as group size k increases, the MSEs associated with unknown parameters are decrease. It is also observed that as the failure proportion m/N increases, the point estimates become even better. Also, the BEs using gamma informative prior are better as they include prior information than MLEs in term of MSEs. The ALs of ACI/HPD credible intervals narrow down as m/N increases, the point estimates become even better. Also, the BEs using gamma informative prior are better as they include prior information than MLEs in term of MSEs. The ALs of ACI/HPD credible intervals narrow down as N increases. As m/N increases, the HPD credible intervals are better than ACIs in respect of ALs. Also, as k increases, the ALs of ACIs increases while the ALs of HPD credible intervals narrows down. Furthermore, in most cases with comparing the CS-I and CS-III, it is clear that the MSEs of the MLEs and BEs for the unknown parameters and reliability characteristics are greater for the CS-III than CS-I. Therefore, we recommend the Bayesian point and interval estimation of the unknown parameters and reliability characteristics of GGD using Gibbs within M-H algorithm.

7 Concluding Remarks

In this paper, the point and interval estimating problems of the unknown parameters and reliability characteristics of GGD is discussed under progressively first-failure-censored samples. We computed MLEs and BEs with SEL, LL and GEL functions. The two-sided ACIs of the unknown parameters and reliability characteristics are constructed using observed Fisher information matrix. It is observed that the BEs cannot be obtained in explicit forms but can be computed numerically. Because of that we have used MCMC techniques such as Gibbs and M-H samplers to generate posterior samples in order to obtain BEs and associated HPD credible intervals. A simulation study was conducted to examine and compare the performance of the proposed methods for different values of N, k, m and different censoring schemes. A numerical example has been presented to illustrate the theoretical results. From the new results, many several works may be obtained as a special cases, such as: (Wu, Hung and Tsai, 2003) if putting $\alpha = 1$ and $\mathbf{R} = (0, 0, \dots, 0)$; (Wu, Chang and Tsai, 2003) if putting $\alpha = k = 1$; (Yarmohammadi and Pazira, 2010) if putting $\mathbf{R} = (0, 0, \dots, 0)$, $k = 1$ and $\beta = 0$; (Madi and Raqab, 2009), (Elkahlout, 2010) if putting $k = 1$ and $\beta = 0$; (Soliman et al., 2012) and (Soliman

and Al Sobhi, 2015) if putting $\alpha = 1$; (Abu-Zinadah, 2014) if putting $\mathbf{R} = (0, 0, \dots, n - m)$ and $k = 1$; (Ghitany, Alqallaf and Balakrishnan, 2014) if putting $\mathbf{R} = (0, 0, \dots, 0)$ and $\alpha = k = 1$; (Ghitany et al., 2014) and (Mohan and Chacko, 2016) if putting $\alpha = k = 1$; (Abu-Zinadah and Bakoban, 2017), (Ahmed, 2015), (Demir and Saraçoğlu, 2015), if putting $k = 1$; and (Mohammed et al., 2017) if putting $\beta = 0$. We hope that the results and methodology discussed in this paper will be beneficial to data analyst and reliability practitioners.

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Table 6: Average estimates of α with their MSEs (in parentheses), when $k = 2$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.9831 (3.2×10^{-4})	0.9930 (1.9×10^{-4})	0.9933 (4.5×10^{-5})	0.9926 (5.5×10^{-5})	0.9932 (4.6×10^{-5})	0.9926 (5.5×10^{-5})
		II	0.9793 (4.7×10^{-4})	1.0060 (3.9×10^{-4})	1.0069 (4.8×10^{-5})	1.0051 (2.6×10^{-5})	1.0067 (4.5×10^{-5})	1.0050 (2.5×10^{-5})
		III	0.9827 (3.4×10^{-4})	0.9966 (1.5×10^{-4})	0.9969 (9.4×10^{-6})	0.9962 (1.4×10^{-5})	0.9969 (9.8×10^{-6})	0.9961 (1.5×10^{-5})
	10	I	0.9760 (6.7×10^{-4})	0.9872 (3.4×10^{-4})	0.9877 (1.5×10^{-4})	0.9868 (1.8×10^{-4})	0.9876 (1.5×10^{-4})	0.9867 (1.8×10^{-4})
		II	0.9653 (1.3×10^{-3})	0.9914 (3.4×10^{-4})	0.9920 (6.3×10^{-5})	0.9907 (8.7×10^{-5})	0.9919 (6.5×10^{-5})	0.9905 (8.9×10^{-5})
		III	0.9659 (1.3×10^{-3})	0.9742 (8.7×10^{-4})	0.9747 (6.4×10^{-4})	0.9736 (6.9×10^{-4})	0.9746 (6.5×10^{-4})	0.9735 (7.0×10^{-4})
	4	I	0.9548 (2.3×10^{-3})	0.9809 (5.5×10^{-4})	0.9813 (3.5×10^{-4})	0.9804 (3.9×10^{-4})	0.9813 (3.5×10^{-4})	0.9803 (3.9×10^{-4})
		II	0.9445 (3.3×10^{-3})	0.9791 (8.6×10^{-4})	0.9802 (3.9×10^{-4})	0.9780 (4.8×10^{-4})	0.9800 (4.0×10^{-4})	0.9778 (4.9×10^{-4})
		III	0.9362 (4.2×10^{-3})	0.9421 (3.7×10^{-3})	0.9431 (3.2×10^{-3})	0.9411 (3.5×10^{-3})	0.9429 (3.3×10^{-3})	0.9408 (3.5×10^{-3})
50	40	I	0.9861 (2.1×10^{-4})	1.0041 (1.2×10^{-4})	1.0044 (1.9×10^{-5})	1.0038 (1.5×10^{-5})	1.0043 (1.9×10^{-5})	1.0038 (1.4×10^{-5})
		II	0.9814 (3.6×10^{-4})	0.9907 (1.3×10^{-4})	0.9908 (8.4×10^{-5})	0.9906 (8.9×10^{-5})	0.9908 (8.5×10^{-5})	0.9905 (8.9×10^{-5})
		III	0.9832 (3.0×10^{-4})	0.9929 (1.3×10^{-4})	0.9932 (4.7×10^{-5})	0.9927 (5.3×10^{-5})	0.9931 (4.7×10^{-5})	0.9927 (5.3×10^{-5})
	25	I	0.9818 (3.6×10^{-4})	0.9943 (3.0×10^{-4})	0.9950 (2.5×10^{-5})	0.9937 (4.0×10^{-5})	0.9949 (2.6×10^{-5})	0.9935 (4.2×10^{-5})
		II	0.9666 (1.2×10^{-3})	0.9798 (4.7×10^{-4})	0.9800 (4.0×10^{-4})	0.9796 (4.2×10^{-4})	0.9799 (4.0×10^{-4})	0.9796 (4.2×10^{-4})
		III	0.9690 (1.0×10^{-3})	0.9839 (3.4×10^{-4})	0.9842 (2.5×10^{-4})	0.9837 (2.7×10^{-4})	0.9841 (2.5×10^{-4})	0.9837 (2.7×10^{-4})
	10	I	0.9678 (1.2×10^{-3})	0.9748 (8.4×10^{-4})	0.9753 (6.1×10^{-4})	0.9742 (6.6×10^{-4})	0.9752 (6.1×10^{-4})	0.9741 (6.7×10^{-4})
		II	0.9444 (3.2×10^{-3})	0.9582 (1.8×10^{-3})	0.9583 (1.7×10^{-3})	0.9580 (1.8×10^{-3})	0.9583 (1.7×10^{-3})	0.9580 (1.8×10^{-3})
		III	0.9378 (3.9×10^{-3})	0.9391 (3.7×10^{-3})	0.9391 (3.7×10^{-3})	0.9390 (3.7×10^{-3})	0.9391 (3.7×10^{-3})	0.9390 (3.7×10^{-3})

Table 7: Average estimates of α with their MSEs (in parentheses), when $k = 5$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.9853	0.9928	0.9931	0.9925	0.9930	0.9924
			(2.3×10^{-4})	(1.6×10^{-4})	(4.8×10^{-5})	(5.7×10^{-5})	(4.9×10^{-5})	(5.8×10^{-5})
		II	0.9814	0.9939	0.9944	0.9934	0.9943	0.9933
			(3.8×10^{-4})	(2.1×10^{-4})	(3.2×10^{-5})	(4.3×10^{-5})	(3.3×10^{-5})	(4.4×10^{-5})
		III	0.9814	0.9844	0.9848	0.9840	0.9847	0.9840
			(3.9×10^{-4})	(3.9×10^{-4})	(2.3×10^{-4})	(2.6×10^{-4})	(2.3×10^{-4})	(2.6×10^{-4})
	10	I	0.9795	1.0169	1.0173	1.0165	1.0172	1.0165
			(5.0×10^{-4})	(4.3×10^{-4})	(3.0×10^{-4})	(2.7×10^{-4})	(3.0×10^{-4})	(2.7×10^{-4})
		II	0.9697	0.9830	0.9837	0.9823	0.9836	0.9821
			(1.0×10^{-3})	(5.7×10^{-4})	(2.6×10^{-4})	(3.1×10^{-4})	(2.7×10^{-4})	(3.2×10^{-4})
		III	0.9655	0.9896	0.9907	0.9884	0.9905	0.9881
			(1.3×10^{-3})	(5.7×10^{-4})	(8.6×10^{-5})	(1.4×10^{-4})	(9.0×10^{-5})	(1.4×10^{-4})
50	4	I	0.9625	0.9935	0.9948	0.9922	0.9946	0.9920
			(1.7×10^{-3})	(5.5×10^{-4})	(2.7×10^{-5})	(6.0×10^{-5})	(3.0×10^{-5})	(6.4×10^{-5})
		II	0.9506	0.9724	0.9742	0.9707	0.9739	0.9703
			(2.7×10^{-3})	(1.5×10^{-3})	(6.7×10^{-4})	(8.6×10^{-4})	(6.8×10^{-4})	(8.8×10^{-4})
		III	0.9373	0.9561	0.9570	0.9552	0.9569	0.9550
			(4.0×10^{-3})	(2.3×10^{-3})	(1.9×10^{-3})	(2.0×10^{-3})	(1.9×10^{-3})	(2.0×10^{-3})
	25	I	0.9884	1.0023	1.0024	1.0022	1.0024	1.0022
			(1.7×10^{-4})	(4.1×10^{-5})	(5.9×10^{-6})	(4.9×10^{-6})	(5.8×10^{-6})	(4.8×10^{-6})
			0.9854	0.9977	0.9979	0.9975	0.9979	0.9975
		II	(2.6×10^{-4})	(8.5×10^{-5})	(4.2×10^{-6})	(6.2×10^{-6})	(4.4×10^{-6})	(6.4×10^{-6})
			0.9879	0.9895	0.9897	0.9893	0.9897	0.9892
			(2.2×10^{-4})	(1.9×10^{-4})	(1.1×10^{-4})	(1.2×10^{-4})	(1.1×10^{-4})	(1.2×10^{-4})
		III	0.9836	0.9994	0.9997	0.9991	0.9996	0.9991
			(2.9×10^{-4})	(9.9×10^{-5})	(1.2×10^{-7})	(7.8×10^{-7})	(1.6×10^{-7})	(8.7×10^{-7})
			0.9705	0.9798	0.9800	0.9797	0.9799	0.9796
		III	(9.1×10^{-4})	(4.7×10^{-4})	(4.0×10^{-4})	(4.1×10^{-4})	(4.0×10^{-4})	(4.2×10^{-4})
			0.9675	0.9932	0.9939	0.9925	0.9938	0.9924
			(1.1×10^{-3})	(3.2×10^{-4})	(3.7×10^{-5})	(5.6×10^{-5})	(3.8×10^{-5})	(5.8×10^{-5})
100	10	I	0.9735	0.9859	0.9864	0.9854	0.9864	0.9853
			(8.3×10^{-3})	(3.9×10^{-4})	(1.8×10^{-4})	(2.1×10^{-4})	(1.9×10^{-4})	(2.2×10^{-4})
		II	0.9516	0.9673	0.9675	0.9671	0.9675	0.9670
			(2.5×10^{-3})	(1.14×10^{-3})	(1.1×10^{-3})	(1.1×10^{-3})	(1.1×10^{-3})	(1.1×10^{-3})
		III	0.9396	0.9482	0.9483	0.9480	0.9483	0.9480
			(3.7×10^{-3})	(2.8×10^{-3})	(2.7×10^{-3})	(2.7×10^{-3})	(2.7×10^{-3})	(2.7×10^{-3})

Table 8: Average estimates of β with their MSEs (in parentheses), when $k = 2$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	1.0836	1.0177	1.0193	1.0161	1.0190	1.0157
			(7.0×10^{-3})	(9.6×10^{-4})	(3.7×10^{-4})	(2.6×10^{-4})	(3.6×10^{-4})	(2.5×10^{-4})
		II	1.0825	1.0636	1.0666	1.0607	1.0658	1.0603
			(6.8×10^{-3})	(5.2×10^{-3})	(4.4×10^{-4})	(3.7×10^{-3})	(4.3×10^{-4})	(3.6×10^{-3})
		III	1.0838	1.0480	1.0505	1.0453	1.0499	1.0499
			(7.0×10^{-3})	(3.3×10^{-3})	(2.6×10^{-3})	(2.1×10^{-3})	(2.5×10^{-3})	(2.0×10^{-3})
	10	I	1.0813	1.0393	1.0414	1.0373	1.0409	1.0371
			(6.6×10^{-3})	(2.4×10^{-3})	(1.7×10^{-3})	(1.4×10^{-3})	(1.7×10^{-3})	(1.4×10^{-3})
		II	1.0758	1.0387	1.0468	1.0301	1.0450	1.0285
			(5.8×10^{-3})	(4.9×10^{-3})	(2.2×10^{-3})	(9.0×10^{-4})	(2.0×10^{-3})	(8.2×10^{-4})
		III	1.0761	1.0680	1.0695	1.0665	1.0691	1.0663
			(5.9×10^{-3})	(5.2×10^{-3})	(4.8×10^{-3})	(4.4×10^{-3})	(4.8×10^{-3})	(4.4×10^{-3})
50	4	I	1.0647	1.0496	1.0513	1.0480	1.0509	1.0478
			(4.5×10^{-3})	(3.1×10^{-3})	(2.6×10^{-3})	(2.3×10^{-3})	(2.6×10^{-3})	(2.3×10^{-3})
		II	1.0541	1.0276	1.0327	1.0226	1.0316	1.0217
			(3.3×10^{-3})	(2.8×10^{-3})	(1.1×10^{-3})	(5.1×10^{-4})	(9.9×10^{-4})	(4.7×10^{-4})
		III	1.0414	1.0244	1.0274	1.0215	1.0267	1.0210
			(2.0×10^{-3})	(1.8×10^{-3})	(7.5×10^{-4})	(4.6×10^{-4})	(7.1×10^{-4})	(4.4×10^{-4})
	25	I	1.0835	1.0147	1.0159	1.0134	1.0156	1.0132
			(7.0×10^{-3})	(7.0×10^{-4})	(2.5×10^{-4})	(1.8×10^{-4})	(2.4×10^{-4})	(1.8×10^{-4})
			1.0824	1.0285	1.0318	1.0247	1.0311	1.0239
		II	(6.8×10^{-3})	(2.2×10^{-3})	(1.0×10^{-3})	(6.1×10^{-4})	(9.7×10^{-4})	(5.7×10^{-4})
			1.0837	1.0527	1.0560	1.0492	1.0552	1.0487
			(7.0×10^{-3})	(4.1×10^{-3})	(3.1×10^{-3})	(2.4×10^{-3})	(3.1×10^{-3})	(2.4×10^{-3})
		III	1.0830	1.0483	1.0490	1.0476	1.0489	1.0475
			(6.9×10^{-3})	(2.6×10^{-3})	(2.4×10^{-3})	(2.3×10^{-3})	(2.4×10^{-3})	(2.3×10^{-3})
			1.0756	1.0465	1.0510	1.0420	1.0499	1.0414
		II	(5.8×10^{-3})	(4.0×10^{-3})	(2.6×10^{-3})	(1.8×10^{-3})	(2.5×10^{-3})	(1.7×10^{-3})
			1.0769	1.0326	1.0353	1.0298	1.0347	1.0294
			(6.0×10^{-3})	(2.2×10^{-3})	(1.2×10^{-3})	(8.9×10^{-4})	(1.2×10^{-3})	(8.6×10^{-4})
10	I	I	1.0765	0.9561	0.9574	0.9547	0.9572	0.9544
			(5.9×10^{-3})	(2.5×10^{-3})	(1.8×10^{-3})	(2.1×10^{-3})	(1.8×10^{-3})	(2.1×10^{-3})
		II	1.0556	1.0336	1.0349	1.0324	1.0346	1.0322
			(3.2×10^{-3})	(1.6×10^{-3})	(1.2×10^{-3})	(1.1×10^{-3})	(1.2×10^{-3})	(1.0×10^{-3})
	III	I	1.0743	1.0272	1.0293	1.0251	1.0288	1.0248
			(2.1×10^{-3})	(1.6×10^{-3})	(8.6×10^{-4})	(6.3×10^{-4})	(8.3×10^{-4})	(6.1×10^{-4})

Table 9: Average estimates of β with their MSEs (in parentheses), when $k = 5$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	1.0836	1.0789	1.0808	1.0771	1.0803	1.0768
			(7.0×10^{-3})	(7.0×10^{-3})	(6.5×10^{-3})	(5.9×10^{-3})	(6.5×10^{-3})	(5.9×10^{-3})
		II	1.0821	0.9686	0.9737	0.9636	0.9728	0.9624
			(6.8×10^{-3})	(3.0×10^{-3})	(6.9×10^{-4})	(1.3×10^{-3})	(7.4×10^{-4})	(1.4×10^{-3})
		III	1.0824	1.0443	1.0453	1.0434	1.0450	1.0433
			(6.8×10^{-3})	(2.3×10^{-3})	(2.1×10^{-3})	(1.9×10^{-3})	(2.0×10^{-3})	(1.9×10^{-3})
	10	I	1.0817	1.0400	1.0450	1.0350	1.0438	1.0342
			(6.7×10^{-3})	(3.6×10^{-3})	(2.0×10^{-3})	(1.2×10^{-3})	(1.9×10^{-3})	(1.2×10^{-3})
		II	1.0771	1.0509	1.0525	1.0494	1.0521	1.0492
			(6.0×10^{-3})	(3.2×10^{-3})	(2.8×10^{-3})	(2.4×10^{-3})	(2.7×10^{-3})	(2.4×10^{-3})
		III	1.0743	1.0228	1.0340	1.0112	1.0316	1.0090
			(5.6×10^{-3})	(5.1×10^{-3})	(1.2×10^{-3})	(1.2×10^{-4})	(9.9×10^{-4})	(8.1×10^{-5})
50	4	I	1.0703	1.0609	1.0642	1.0576	1.0634	1.0572
			(5.2×10^{-3})	(5.0×10^{-3})	(4.1×10^{-3})	(3.3×10^{-3})	(4.0×10^{-3})	(3.3×10^{-3})
		II	1.0597	1.0315	1.0373	1.0252	1.0360	1.0241
			(3.9×10^{-3})	(3.4×10^{-3})	(1.4×10^{-3})	(6.4×10^{-4})	(1.3×10^{-3})	(5.8×10^{-4})
		III	1.0408	1.0496	1.0509	1.0483	1.0506	1.0482
			(2.0×10^{-3})	(3.0×10^{-3})	(2.6×10^{-3})	(2.3×10^{-3})	(2.6×10^{-3})	(2.3×10^{-3})
	25	I	1.0732	0.9740	0.9833	0.9649	0.9816	0.9629
			(6.1×10^{-3})	(4.4×10^{-3})	(2.8×10^{-4})	(1.2×10^{-3})	(3.4×10^{-4})	(1.4×10^{-3})
			1.0707	1.0028	1.0037	1.0020	1.0035	1.0018
		II	(5.8×10^{-3})	(3.5×10^{-4})	(1.4×10^{-5})	(3.8×10^{-6})	(1.2×10^{-5})	(3.2×10^{-6})
			1.0599	0.9882	0.9896	0.9867	0.9893	0.9863
			(4.9×10^{-3})	(7.2×10^{-4})	(1.1×10^{-4})	(1.8×10^{-4})	(1.1×10^{-4})	(1.9×10^{-4})
		III	1.0828	1.0459	1.0484	1.0432	1.0478	1.0428
			(6.9×10^{-3})	(3.1×10^{-3})	(2.3×10^{-3})	(1.9×10^{-3})	(2.3×10^{-3})	(1.8×10^{-3})
			1.0773	0.9674	0.9757	0.9595	0.9742	0.9576
		III	(6.0×10^{-3})	(4.3×10^{-3})	(5.9×10^{-4})	(1.6×10^{-3})	(6.7×10^{-4})	(1.8×10^{-3})
			1.0745	1.0572	1.0621	1.0521	1.0609	1.0514
			(5.6×10^{-3})	(5.3×10^{-3})	(3.9×10^{-3})	(2.7×10^{-3})	(3.7×10^{-3})	(2.6×10^{-3})
100	10	I	1.0786	1.0163	1.0198	1.0126	1.0191	1.0119
			(6.2×10^{-3})	(1.7×10^{-3})	(3.9×10^{-3})	(1.6×10^{-4})	(3.6×10^{-4})	(1.4×10^{-4})
		II	1.0623	1.0323	1.0337	1.0309	1.0334	1.0307
			(4.0×10^{-3})	(1.6×10^{-3})	(1.1×10^{-3})	(9.5×10^{-4})	(1.1×10^{-3})	(9.4×10^{-4})
		III	1.0422	1.0186	1.0200	1.0171	1.0197	1.0168
			(1.9×10^{-3})	(9.2×10^{-4})	(4.0×10^{-4})	(2.9×10^{-4})	(3.9×10^{-4})	(2.8×10^{-4})

Table 10: Average estimates of λ with their MSEs (in parentheses), when $k = 2$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.0038	0.1091	0.1116	0.1068	0.1247	0.0828
			(9.6×10^{-3})	(1.0×10^{-3})	(1.3×10^{-4})	(4.6×10^{-5})	(6.1×10^{-4})	(2.7×10^{-4})
		II	0.0039	0.1098	0.1111	0.1085	0.1180	0.0950
			(9.2×10^{-3})	(6.1×10^{-4})	(1.2×10^{-4})	(7.2×10^{-5})	(3.2×10^{-4})	(2.5×10^{-5})
		III	0.0042	0.1069	0.1090	0.1051	0.1212	0.0905
			(9.2×10^{-3})	(8.2×10^{-4})	(8.0×10^{-5})	(2.6×10^{-5})	(4.5×10^{-4})	(8.9×10^{-5})
	10	I	0.0033	0.0868	0.0874	0.0861	0.0918	0.0783
			(9.7×10^{-3})	(4.1×10^{-4})	(1.6×10^{-4})	(1.9×10^{-4})	(6.7×10^{-5})	(4.7×10^{-4})
		II	0.0039	0.0709	0.0717	0.0701	0.0782	0.0541
			(9.2×10^{-3})	(1.2×10^{-3})	(8.0×10^{-4})	(8.9×10^{-4})	(4.8×10^{-4})	(2.1×10^{-3})
		III	0.0046	0.1825	0.1868	0.1782	0.1990	0.1503
			(9.1×10^{-3})	(8.5×10^{-3})	(7.5×10^{-3})	(6.1×10^{-3})	(9.8×10^{-3})	(2.5×10^{-3})
50	4	I	0.0023	0.0622	0.0629	0.0616	0.0700	0.0499
			(9.6×10^{-3})	(1.7×10^{-3})	(1.4×10^{-3})	(1.5×10^{-3})	(9.0×10^{-4})	(2.5×10^{-3})
		II	0.0027	0.0551	0.0571	0.0531	0.0745	0.0079
			(9.5×10^{-3})	(2.8×10^{-3})	(1.8×10^{-3})	(2.2×10^{-3})	(6.5×10^{-4})	(8.5×10^{-3})
		III	0.0036	0.1004	0.1034	0.0977	0.1225	0.0791
			(9.3×10^{-3})	(1.2×10^{-3})	(1.2×10^{-5})	(5.4×10^{-6})	(5.1×10^{-4})	(4.4×10^{-4})
	25	I	00.0047	0.0816	0.0820	0.0811	0.0855	0.0751
			(9.1×10^{-3})	(5.1×10^{-4})	(3.3×10^{-4})	(3.6×10^{-4})	(2.1×10^{-4})	(6.2×10^{-4})
			0.0052	0.1027	0.1034	0.1019	0.1079	0.0926
		II	(9.0×10^{-3})	(3.0×10^{-4})	(1.3×10^{-5})	(3.7×10^{-6})	(6.2×10^{-5})	(5.5×10^{-5})
			0.0051	0.1186	0.1190	0.1174	0.1265	0.1064
			(9.0×10^{-3})	(8.4×10^{-4})	(4.0×10^{-4})	(3.0×10^{-4})	(7.0×10^{-4})	(4.1×10^{-5})
		III	0.0045	0.0863	0.0872	0.0856	0.0933	0.0763
			(9.1×10^{-3})	(5.0×10^{-4})	(1.6×10^{-4})	(2.1×10^{-4})	(4.4×10^{-5})	(5.6×10^{-4})
			0.0059	0.1817	0.1863	0.1766	0.1975	0.1367
		III	(8.9×10^{-3})	(8.6×10^{-3})	(7.4×10^{-3})	(5.9×10^{-3})	(9.5×10^{-3})	(1.4×10^{-3})
			0.0071	0.1692	0.1701	0.1683	0.1731	0.1628
			(8.6×10^{-3})	(5.1×10^{-3})	(4.9×10^{-3})	(4.7×10^{-3})	(5.4×10^{-3})	(3.9×10^{-3})
10	I	I	0.0035	0.1569	0.1586	0.1552	0.1650	0.1426
			(9.3×10^{-3})	(3.9×10^{-3})	(3.4×10^{-3})	(3.0×10^{-3})	(4.2×10^{-3})	(1.8×10^{-3})
		II	0.0052	0.1547	0.1573	0.1520	0.1661	0.1288
			(9.0×10^{-3})	(4.1×10^{-3})	(3.3×10^{-3})	(2.7×10^{-3})	(4.4×10^{-3})	(8.3×10^{-4})
		III	0.0073	0.1643	0.1656	0.1631	0.1697	0.1540
			(8.6×10^{-3})	(4.6×10^{-3})	(4.3×10^{-3})	(4.0×10^{-3})	(4.9×10^{-3})	(2.9×10^{-3})

Table 11: Average estimates of λ with their MSEs (in parentheses), when $k = 5$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.0048	0.0561	0.0565	0.0558	0.0598	0.0467
			(9.1×10^{-3})	(2.0×10^{-3})	(1.9×10^{-3})	(2.0×10^{-3})	(1.6×10^{-3})	(2.8×10^{-3})
		II	0.0055	0.1021	0.1038	0.1005	0.1139	0.0838
			(8.9×10^{-3})	(6.5×10^{-4})	(1.4×10^{-5})	(2.5×10^{-7})	(1.9×10^{-4})	(2.6×10^{-4})
		III	0.0057	0.1223	0.1235	0.1211	0.1291	0.1101
			(8.9×10^{-3})	(9.6×10^{-4})	(5.5×10^{-4})	(4.5×10^{-4})	(8.5×10^{-4})	(1.0×10^{-4})
	10	I	0.0048	0.0855	0.0867	0.0843	0.0957	0.0662
			(9.1×10^{-3})	(6.7×10^{-4})	(1.8×10^{-4})	(2.5×10^{-4})	(1.8×10^{-5})	(1.1×10^{-3})
		II	0.0057	0.0948	0.0957	0.0938	0.1020	0.0836
			(8.9×10^{-3})	(3.9×10^{-3})	(1.9×10^{-5})	(3.8×10^{-5})	(3.9×10^{-6})	(2.7×10^{-4})
		III	0.0071	0.1577	0.1596	0.1557	0.1662	0.1396
			(8.6×10^{-3})	(4.1×10^{-3})	(3.6×10^{-3})	(3.1×10^{-3})	(4.4×10^{-3})	(1.6×10^{-3})
50	4	I	0.0043	0.1022	0.1039	0.1005	0.1138	0.0820
			(9.2×10^{-3})	(6.7×10^{-4})	(1.5×10^{-5})	(2.6×10^{-7})	(1.9×10^{-4})	(3.2×10^{-4})
		II	0.0051	0.1073	0.1115	0.1028	0.1285	0.0464
			(9.0×10^{-3})	(1.8×10^{-3})	(1.3×10^{-4})	(7.6×10^{-6})	(8.1×10^{-4})	(2.9×10^{-3})
		III	0.0067	0.1360	0.1421	0.1298	0.1626	0.0794
			(8.7×10^{-3})	(3.8×10^{-3})	(1.8×10^{-3})	(8.9×10^{-4})	(3.9×10^{-3})	(4.3×10^{-4})
	25	I	0.0165	0.0908	0.0919	0.0898	0.0994	0.0795
			(7.9×10^{-3})	(4.9×10^{-4})	(6.6×10^{-5})	(1.0×10^{-4})	(3.1×10^{-7})	(4.2×10^{-4})
			0.0187	0.1029	0.1034	0.1025	0.1062	0.0974
		II	(7.7×10^{-3})	(1.8×10^{-4})	(1.1×10^{-5})	(6.1×10^{-6})	(3.9×10^{-5})	(6.8×10^{-6})
			0.0320	0.1482	0.1506	0.1460	0.1609	0.1350
			(6.4×10^{-3})	(3.2×10^{-3})	(2.6×10^{-3})	(2.1×10^{-3})	(3.7×10^{-3})	(1.2×10^{-3})
		III	0.0054	0.1003	0.1020	0.0985	0.1122	0.0702
			(9.0×10^{-3})	(6.8×10^{-4})	(4.0×10^{-6})	(2.1×10^{-6})	(1.5×10^{-4})	(8.9×10^{-4})
			0.0068	0.1187	0.1204	0.1169	0.1285	0.0968
		II	(8.7×10^{-3})	(1.0×10^{-3})	(4.2×10^{-4})	(2.9×10^{-4})	(8.1×10^{-4})	(1.1×10^{-5})
			0.0088	0.1774	0.1805	0.1740	0.1892	0.1484
			(8.4×10^{-3})	(7.3×10^{-3})	(6.5×10^{-3})	(5.5×10^{-3})	(8.0×10^{-3})	(2.4×10^{-3})
10	10	I	0.0052	0.0703	0.0709	0.0697	0.0766	0.0581
			(9.0×10^{-3})	(1.1×10^{-3})	(8.5×10^{-4})	(9.2×10^{-4})	(5.5×10^{-4})	(1.8×10^{-3})
		II	0.0070	0.0954	0.0964	0.0944	0.1021	0.0804
			(8.7×10^{-3})	(4.1×10^{-4})	(1.3×10^{-5})	(3.1×10^{-5})	(4.4×10^{-6})	(3.8×10^{-4})
		III	0.0113	0.1620	0.1638	0.1600	0.1700	0.1445
			(7.9×10^{-3})	(4.6×10^{-3})	(4.1×10^{-3})	(3.6×10^{-3})	(4.9×10^{-3})	(2.0×10^{-3})

Table 12: Average estimates of $S(0.5)$ with their MSEs (in parentheses), when $k = 2$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.9972	0.9304	0.9312	0.9296	0.9311	0.9294
			(3.6×10^{-3})	(3.6×10^{-4})	(3.6×10^{-5})	(5.7×10^{-5})	(3.7×10^{-5})	(6.1×10^{-5})
		II	0.9971	0.9315	0.9319	0.9311	0.9318	0.9309
			(3.6×10^{-3})	(1.9×10^{-4})	(2.8×10^{-5})	(3.8×10^{-5})	(2.9×10^{-5})	(3.9×10^{-5})
		III	0.9970	0.9316	0.9324	0.9308	0.9323	0.9305
			(3.6×10^{-3})	(3.5×10^{-4})	(2.3×10^{-5})	(4.1×10^{-5})	(2.4×10^{-5})	(4.5×10^{-5})
	10	I	0.9975	0.9428	0.9430	0.9426	0.9429	0.9425
			(3.6×10^{-3})	(1.1×10^{-4})	(3.4×10^{-5})	(2.9×10^{-5})	(3.3×10^{-5})	(2.8×10^{-5})
		II	0.9969	0.9533	0.9537	0.9529	0.9536	0.9528
			(3.6×10^{-3})	(4.0×10^{-4})	(2.7×10^{-4})	(2.5×10^{-4})	(2.7×10^{-4})	(2.5×10^{-4})
		III	0.9963	0.8802	0.8818	0.8786	0.8817	0.8780
			(3.5×10^{-3})	(3.9×10^{-3})	(3.1×10^{-3})	(3.4×10^{-3})	(3.1×10^{-3})	(3.5×10^{-3})
50	4	I	0.9980	0.9575	0.9578	0.9572	0.9577	0.9571
			(3.7×10^{-3})	(5.2×10^{-4})	(4.2×10^{-4})	(4.0×10^{-4})	(4.2×10^{-4})	(4.0×10^{-4})
		II	0.9975	0.9615	0.9626	0.9604	0.9625	0.9602
			(3.6×10^{-3})	(1.0×10^{-3})	(6.5×10^{-4})	(5.4×10^{-4})	(6.4×10^{-4})	(5.3×10^{-4})
		III	0.9966	0.9254	0.9271	0.9235	0.9269	0.9229
			(3.5×10^{-3})	(8.4×10^{-4})	(1.0×10^{-4})	(1.9×10^{-4})	(1.1×10^{-4})	(2.0×10^{-4})
	25	I	0.9967	0.9488	0.9490	0.9486	0.9491	0.9486
			(3.5×10^{-3})	(2.1×10^{-4})	(1.4×10^{-4})	(1.3×10^{-4})	(1.4×10^{-4})	(1.3×10^{-4})
			0.9962	0.9334	0.9338	0.9931	0.9337	0.9930
		II	(3.5×10^{-3})	(1.4×10^{-4})	(1.2×10^{-5})	(1.7×10^{-5})	(1.2×10^{-5})	(1.7×10^{-5})
			0.9963	0.9237	0.9241	0.9232	0.9241	0.9231
			(3.5×10^{-3})	(3.6×10^{-4})	(1.7×10^{-4})	(2.0×10^{-4})	(1.7×10^{-4})	(2.0×10^{-4})
		III	0.9967	0.9440	0.9443	0.9437	0.9443	0.9436
			(3.6×10^{-3})	(1.6×10^{-4})	(5.1×10^{-5})	(4.3×10^{-5})	(5.0×10^{-5})	(4.2×10^{-5})
			0.9953	0.8829	0.8846	0.8812	0.8844	0.8807
		III	(3.4×10^{-3})	(3.6×10^{-3})	(2.8×10^{-3})	(3.1×10^{-3})	(2.8×10^{-3})	(3.2×10^{-3})
			0.9945	0.8913	0.8917	0.8909	0.8917	0.8908
			(3.3×10^{-3})	(2.3×10^{-3})	(2.1×10^{-3})	(2.1×10^{-3})	(2.1×10^{-3})	(2.2×10^{-3})
10	10	I	0.9972	0.8986	0.8993	0.8979	0.8992	0.8977
			(3.6×10^{-3})	(1.8×10^{-3})	(1.4×10^{-3})	(1.5×10^{-3})	(1.4×10^{-3})	(1.6×10^{-3})
		II	0.9954	0.8940	0.8951	0.8930	0.8950	0.8926
			(3.4×10^{-3})	(2.3×10^{-3})	(1.8×10^{-3})	(2.0×10^{-3})	(1.8×10^{-3})	(2.0×10^{-3})
		III	0.9935	0.8831	0.8836	0.8826	0.8836	0.8825
			(3.2×10^{-3})	(3.1×10^{-3})	(2.9×10^{-3})	(3.0×10^{-3})	(2.9×10^{-3})	(3.0×10^{-3})

Table 13: Average estimates of $S(0.5)$ with their MSEs (in parentheses), when $k = 5$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.9966	0.9626	0.9628	0.962	5 0.9627	0.9624
			(3.5×10^{-3})	(7.0×10^{-4})	(6.5×10^{-4})	(6.4×10^{-4})	(6.5×10^{-4})	(6.4×10^{-4})
		II	0.9960	0.9354	0.9360	0.9347	0.9359	0.9345
			(3.5×10^{-3})	(2.6×10^{-4})	(1.4×10^{-6})	(6.2×10^{-6})	(1.6×10^{-6})	(7.2×10^{-6})
		III	0.9958	0.9196	0.9202	0.9191	0.9201	0.9189
			(3.4×10^{-3})	(5.3×10^{-4})	(2.9×10^{-4})	(3.3×10^{-4})	(2.9×10^{-4})	(3.4×10^{-4})
	10	I	0.9965	0.9483	0.9487	0.9479	0.9486	0.9478
			(3.5×10^{-3})	(2.7×10^{-4})	(1.3×10^{-4})	(1.2×10^{-4})	(1.3×10^{-4})	(1.1×10^{-4})
		II	0.9956	0.9368	0.9371	0.9365	0.9371	0.9364
			(3.4×10^{-3})	(1.3×10^{-4})	(1.4×10^{-9})	(5.0×10^{-7})	(7.1×10^{-9})	(6.4×10^{-7})
		III	0.9944	0.9002	0.9007	0.8997	0.9006	0.8995
			(3.3×10^{-3})	(1.6×10^{-3})	(1.3×10^{-3})	(1.4×10^{-3})	(1.3×10^{-3})	(1.4×10^{-3})
50	4	I	0.9965	0.9338	0.9344	0.9331	0.9343	0.9329
			(3.5×10^{-3})	(2.7×10^{-4})	(7.6×10^{-6})	(1.7×10^{-5})	(8.1×10^{-6})	(1.8×10^{-5})
		II	0.9956	0.9285	0.9299	0.9271	0.9297	0.9267
			(3.4×10^{-3})	(6.3×10^{-4})	(5.3×10^{-5})	(1.0×10^{-4})	(5.6×10^{-5})	(1.1×10^{-4})
		III	0.9940	0.9060	0.9082	0.9038	0.9080	0.9031
			(3.2×10^{-3})	(1.9×10^{-3})	(8.4×10^{-4})	(1.1×10^{-3})	(8.5×10^{-4})	(1.2×10^{-3})
	25	I	0.9892	0.9438	0.9441	0.9434	0.9441	0.9433
			(3.1×10^{-3})	(1.7×10^{-4})	(4.8×10^{-5})	(3.9×10^{-5})	(4.7×10^{-5})	(3.8×10^{-5})
			0.9878	0.9350	0.9352	0.9348	0.9351	0.9347
		II	(3.0×10^{-3})	(7.8×10^{-5})	(4.1×10^{-6})	(5.9×10^{-6})	(4.2×10^{-6})	(6.2×10^{-6})
			0.9795	0.9064	0.9072	0.9056	0.9072	0.9053
			(2.5×10^{-3})	(1.3×10^{-3})	(9.0×10^{-4})	(9.9×10^{-4})	(9.0×10^{-4})	(1.0×10^{-3})
		III	0.9961	0.9362	0.9369	0.9355	0.9368	0.9353
			(3.5×10^{-3})	(2.7×10^{-4})	(7.6×10^{-8})	(2.8×10^{-6})	(1.1×10^{-7})	(3.5×10^{-6})
			0.9948	0.9230	0.9235	0.9224	0.9235	0.9222
			(3.3×10^{-3})	(4.2×10^{-4})	(1.9×10^{-4})	(2.2×10^{-4})	(1.9×10^{-4})	(2.2×10^{-4})
		III	0.9932	0.8887	0.8896	0.8879	0.8895	0.8877
			(3.2×10^{-3})	(2.7×10^{-3})	(2.3×10^{-3})	(2.4×10^{-3})	(2.3×10^{-3})	(2.5×10^{-3})
10	10	I	0.9960	0.9533	0.9536	0.9531	0.9535	0.9531
			(3.5×10^{-3})	(3.5×10^{-4})	(2.7×10^{-4})	(2.5×10^{-4})	(2.7×10^{-4})	(2.5×10^{-4})
		II	0.9941	0.9338	0.9342	0.9333	0.9341	0.9332
			(3.3×10^{-3})	(1.8×10^{-4})	(8.9×10^{-6})	(1.5×10^{-5})	(9.3×10^{-6})	(1.6×10^{-5})
		III	0.9902	0.8872	0.8881	0.8864	0.8880	0.8861
			(2.8×10^{-3})	(2.8×10^{-3})	(2.4×10^{-3})	(2.6×10^{-3})	(2.4×10^{-3})	(2.6×10^{-3})

Table 14: Average estimates of $H(0.5)$ with their MSEs (in parentheses), when $k = 2$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.0070	0.1834	0.1893	0.1776	0.2058	0.1423
			(2.5×10^{-2})	(2.7×10^{-3})	(6.0×10^{-4})	(1.6×10^{-4})	(1.7×10^{-3})	(5.1×10^{-4})
		II	0.0074	0.1843	0.1873	0.1813	0.1963	0.1644
			(2.5×10^{-2})	(1.6×10^{-3})	(5.0×10^{-4})	(2.7×10^{-4})	(9.9×10^{-4})	(2.5×10^{-7})
		III	0.0078	0.1817	0.1875	0.1764	0.2057	0.1534
			(2.5×10^{-2})	(2.5×10^{-3})	(5.1×10^{-4})	(1.3×10^{-4})	(1.7×10^{-3})	(1.3×10^{-4})
	10	I	0.0063	0.1496	0.1511	0.1481	0.1569	0.1370
			(2.5×10^{-2})	(8.3×10^{-4})	(1.9×10^{-4})	(2.8×10^{-4})	(6.4×10^{-5})	(7.8×10^{-4})
		II	0.0078	0.1218	0.1242	0.1194	0.1347	0.0914
			(2.5×10^{-2})	(2.8×10^{-3})	(1.7×10^{-3})	(2.1×10^{-3})	(9.1×10^{-4})	(5.4×10^{-3})
		III	0.0092	0.3232	0.3366	0.3101	0.3529	0.2705
			(2.4×10^{-2})	(3.0×10^{-2})	(2.9×10^{-2})	(2.1×10^{-2})	(3.5×10^{-2})	(1.1×10^{-2})
50	40	I	0.0049	0.1099	0.1118	0.1080	0.1229	0.0910
			(2.6×10^{-2})	(3.8×10^{-3})	(2.8×10^{-3})	(3.2×10^{-3})	(1.8×10^{-3})	(5.5×10^{-3})
		II	0.0061	0.0987	0.1060	0.0916	0.1372	0.0137
			(2.5×10^{-2})	(7.3×10^{-3})	(3.5×10^{-3})	(5.4×10^{-3})	(7.7×10^{-4})	(2.3×10^{-2})
		III	0.0082	0.1876	0.1992	0.1781	0.2314	0.1472
			(2.5×10^{-2})	(4.7×10^{-3})	(1.2×10^{-3})	(1.7×10^{-4})	(4.4×10^{-3})	(3.1×10^{-4})
	25	I	0.0085	0.1344	0.1357	0.1331	0.1418	0.1227
			(2.6×10^{-2})	(1.4×10^{-3})	(8.5×10^{-4})	(1.0×10^{-3})	(5.3×10^{-4})	(1.8×10^{-3})
			0.0097	0.1749	0.1771	0.1726	0.1841	0.1572
			(2.4×10^{-2})	(9.8×10^{-4})	(1.5×10^{-4})	(6.0×10^{-5})	(3.7×10^{-4})	(5.9×10^{-5})
		II	0.0094	0.2030	0.2062	0.1999	0.2146	0.1845
			(2.4×10^{-2})	(2.7×10^{-3})	(1.7×10^{-3})	(1.2×10^{-3})	(2.5×10^{-3})	(3.9×10^{-4})
			0.0083	0.1474	0.1496	0.1453	0.1581	0.1319
			(2.5×10^{-2})	(1.1×10^{-3})	(2.3×10^{-4})	(3.8×10^{-4})	(4.6×10^{-5})	(1.1×10^{-3})
		III	0.0117	0.3164	0.3294	0.3006	0.3436	0.2371
			(2.4×10^{-2})	(2.9×10^{-2})	(2.7×10^{-2})	(1.8×10^{-2})	(3.2×10^{-2})	(5.2×10^{-3})
			0.0139	0.2908	0.2935	0.2878	0.2982	0.2786
			(2.3×10^{-2})	(1.7×10^{-2})	(1.7×10^{-2})	(1.5×10^{-2})	(1.8×10^{-2})	(1.3×10^{-2})
10	10	I	0.0071	0.2634	0.2682	0.2586	0.2768	0.2404
			(2.5×10^{-2})	(1.2×10^{-2})	(1.1×10^{-2})	(8.8×10^{-3})	(1.3×10^{-2})	(5.7×10^{-3})
		II	0.0112	0.2771	0.2848	0.2688	0.2966	0.2342
			(2.4×10^{-2})	(1.6×10^{-2})	(1.4×10^{-2})	(1.1×10^{-2})	(1.7×10^{-2})	(4.8×10^{-3})
		III	0.0159	0.3012	0.3045	0.2977	0.3093	0.2857
			(2.2×10^{-2})	(2.0×10^{-2})	(1.9×10^{-2})	(1.8×10^{-2})	(2.1×10^{-2})	(1.5×10^{-2})

Table 15: Average estimates of $H(0.5)$ with their MSEs (in parentheses), when $k = 5$.

N	m	CS	MLE	SEL	LL		GEL	
					$h = -5$	$h = 5$	$q = -5$	$q = 5$
20	16	I	0.0088	0.0980	0.0989	0.0971	0.1041	0.0806
			(2.4×10^{-2})	(4.8×10^{-3})	(4.4×10^{-3})	(4.6×10^{-3})	(3.7×10^{-3})	(7.1×10^{-3})
		II	0.0103	0.1679	0.1725	0.1635	0.1870	0.1351
			(2.4×10^{-2})	(1.8×10^{-3})	(5.8×10^{-5})	(1.9×10^{-6})	(4.9×10^{-4})	(8.9×10^{-4})
		III	0.0107	0.2123	0.2163	0.2083	0.2256	0.1872
			(2.4×10^{-2})	(3.8×10^{-3})	(2.6×10^{-3})	(1.9×10^{-3})	(3.7×10^{-3})	(5.0×10^{-4})
	10	I	0.0090	0.1387	0.1417	0.1359	0.1545	0.1097
			(2.4×10^{-2})	(1.8×10^{-3})	(5.4×10^{-4})	(8.4×10^{-4})	(1.1×10^{-4})	(3.0×10^{-3})
		II	0.0111	0.1655	0.1680	0.1631	0.1765	0.1487
			(2.4×10^{-2})	(9.6×10^{-4})	(9.5×10^{-6})	(3.1×10^{-6})	(1.4×10^{-4})	(2.6×10^{-4})
		III	0.0141	0.2667	0.2706	0.2625	0.2773	0.2442
			(2.3×10^{-2})	(1.2×10^{-2})	(1.1×10^{-2})	(9.5×10^{-3})	(1.3×10^{-2})	(6.3×10^{-3})
50	4	I	0.0088	0.1760	0.1809	0.1712	0.1961	0.1403
			(2.4×10^{-2})	(2.1×10^{-3})	(2.6×10^{-4})	(4.0×10^{-5})	(9.8×10^{-4})	(6.1×10^{-4})
		II	0.0109	0.1867	0.1965	0.1757	0.2172	0.0926
			(2.4×10^{-2})	(4.7×10^{-3})	(1.0×10^{-3})	(1.2×10^{-4})	(2.7×10^{-3})	(5.2×10^{-3})
		III	0.0145	0.2462	0.2633	0.2289	0.2900	0.1573
			(2.3×10^{-2})	(1.4×10^{-3})	(9.7×10^{-3})	(4.1×10^{-3})	(1.6×10^{-2})	(5.8×10^{-5})
	25	I	0.0280	0.1464	0.1485	0.1443	0.1573	0.1315
			(2.1×10^{-2})	(1.2×10^{-3})	(2.7×10^{-4})	(4.2×10^{-4})	(5.7×10^{-5})	(1.1×10^{-3})
			0.0318	0.1707	0.1720	0.1694	0.1763	0.1610
			(2.1×10^{-2})	(5.3×10^{-4})	(5.0×10^{-5})	(2.1×10^{-5})	(1.3×10^{-4})	(1.5×10^{-5})
		III	0.0536	0.2471	0.2538	0.2413	0.2682	0.2682
			(1.7×10^{-2})	(9.3×10^{-3})	(7.9×10^{-3})	(5.8×10^{-3})	(1.1×10^{-2})	(1.1×10^{-2})
		I	0.0099	0.1692	0.1740	0.1644	0.1885	0.1160
			(2.4×10^{-2})	(1.9×10^{-3})	(8.3×10^{-5})	(2.5×10^{-7})	(5.6×10^{-4})	(2.4×10^{-3})
		II	0.0131	0.1986	0.2023	0.1949	0.2119	0.1704
			(2.3×10^{-2})	(2.6×10^{-3})	(1.4×10^{-3})	(9.0×10^{-4})	(2.2×10^{-3})	(3.1×10^{-5})
10	10	III	0.0171	0.3025	0.3089	0.2955	0.3175	0.2670
			(2.2×10^{-2})	(2.2×10^{-2})	(2.1×10^{-2})	(1.7×10^{-2})	(2.3×10^{-2})	(1.0×10^{-2})
		I	0.0100	0.1204	0.1220	0.1189	0.1300	0.0974
			(2.4×10^{-2})	(2.6×10^{-3})	(1.8×10^{-3})	(2.1×10^{-3})	(1.2×10^{-3})	(4.6×10^{-3})
		II	0.0146	0.1705	0.1735	0.1675	0.1825	0.1456
			(2.3×10^{-2})	(1.2×10^{-3})	(7.4×10^{-5})	(6.7×10^{-6})	(3.1×10^{-4})	(3.7×10^{-4})
	4	III	0.0238	0.2918	0.2974	0.2859	0.3057	0.2618
			(2.0×10^{-2})	(1.8×10^{-2})	(1.8×10^{-2})	(1.5×10^{-2})	(2.0×10^{-2})	(9.4×10^{-3})

Table 16: The ALs of 95% ACIs for α , β , λ , $S(0.5)$ and $H(0.5)$.

N	m	CS	α		β		λ		$S(0.5)$		$H(0.5)$	
			$k =$	$k = 5$	$k = 2$	$k = 5$	$k = 2$	$k = 5$	$k = 2$	$k = 5$	$k = 2$	$k = 5$
			2									
20	16	I	0.0661	0.0575	0.3281	0.3279	0.3772	0.3731	0.2354	0.2327	0.6187	0.6118
		II	0.0810	0.0729	0.3233	0.3219	0.3766	0.3704	0.2348	0.2304	0.6173	0.6060
		III	0.0678	0.0728	0.3287	0.3230	0.3756	0.3695	0.2343	0.2298	0.6158	0.6045
	10	I	0.0941	0.0802	0.3186	0.3202	0.3792	0.3733	0.2365	0.2324	0.6216	0.6111
		II	0.1360	0.1187	0.2971	0.3021	0.3767	0.3697	0.2339	0.2289	0.6156	0.6026
		III	0.1339	0.1351	0.2982	0.2914	0.3788	0.3640	0.2318	0.2242	0.6101	0.5910
	4	I	0.1772	0.1471	0.2536	0.2757	0.3830	0.3751	0.2384	0.2324	0.6269	0.6118
		II	0.2175	0.1935	0.2121	0.2339	0.3813	0.3718	0.2365	0.2289	0.6225	0.6036
		III	0.2502	0.2457	0.1622	0.1598	0.3777	0.3658	0.2329	0.2227	0.6140	0.5894
50	40	I	0.0544	0.0456	0.3274	0.2869	0.3737	0.3273	0.2331	0.2040	0.6128	0.5364
		II	0.0730	0.0571	0.3230	0.2772	0.3717	0.3186	0.2313	0.1983	0.6084	0.5216
		III	0.0660	0.0475	0.3282	0.2346	0.3719	0.2666	0.2317	0.1657	0.6091	0.4360
	25	I	0.0712	0.0644	0.3253	0.3247	0.3745	0.3709	0.2334	0.2310	0.6136	0.6074
		II	0.1309	0.1158	0.2965	0.3032	0.3688	0.3653	0.2280	0.2259	0.6005	0.5949
		III	0.1216	0.1274	0.3016	0.2920	0.3640	0.3574	0.2246	0.2196	0.5919	0.5793
	10	I	0.1260	0.1039	0.2999	0.3080	0.3782	0.3717	0.2352	0.2307	0.6186	0.6071
		II	0.2181	0.1897	0.2180	0.2442	0.3716	0.3644	0.2282	0.2229	0.6022	0.5889
		III	0.2438	0.2367	0.1714	0.1653	0.3633	0.3476	0.2206	0.2079	0.5841	0.5531

Table 17: The ALs of 95% HPD credible intervals for α , β , λ , $S(0.5)$ and $H(0.5)$.

N	m	CS	α		β		λ		$S(0.5)$		$H(0.5)$	
			$k=2$	$k=5$	$k=2$	$k=5$	$k=2$	$k=5$	$k=2$	$k=5$	$k=2$	$k=5$
20	16	I	0.0404	0.0382	0.0908	0.0978	0.1160	0.0418	0.0660	0.0262	0.1817	0.0701
		II	0.0709	0.0543	0.1151	0.1476	0.0788	0.0971	0.0469	0.0588	0.1275	0.1568
		III	0.0440	0.0404	0.1258	0.0786	0.0999	0.0770	0.0636	0.0536	0.1684	0.1445
	10	I	0.0496	0.0433	0.1144	0.1538	0.0523	0.0923	0.0310	0.0510	0.0845	0.1451
		II	0.0566	0.0654	0.1799	0.0871	0.0679	0.0695	0.0429	0.0418	0.1129	0.1148
		III	0.0496	0.0766	0.0895	0.2071	0.1582	0.1018	0.0902	0.0487	0.2674	0.1394
	4	I	0.0538	0.0817	0.1094	0.1248	0.0599	0.0977	0.0373	0.0644	0.1001	0.1762
		II	0.0617	0.0808	0.1462	0.1618	0.0937	0.1293	0.0666	0.0773	0.1737	0.2108
		III	0.0688	0.0651	0.1086	0.0828	0.1183	0.1729	0.0911	0.1067	0.2262	0.2976
50	40	I	0.0367	0.0227	0.0799	0.1680	0.0501	0.0722	0.0338	0.0409	0.0879	0.1049
		II	0.0252	0.0372	0.1522	0.0672	0.0681	0.0567	0.0444	0.0361	0.1206	0.0945
		III	0.0286	0.0301	0.1245	0.0889	0.0820	0.1072	0.0485	0.0662	0.1287	0.1835
	25	I	0.0606	0.0358	0.0629	0.1288	0.0649	0.1034	0.0377	0.0640	0.1042	0.1704
		II	0.0254	0.0285	0.1296	0.2022	0.1470	0.0964	0.0873	0.0554	0.2543	0.1440
		III	0.0318	0.0511	0.1233	0.1619	0.0680	0.1260	0.0438	0.0656	0.1211	0.1922
	10	I	0.0472	0.0485	0.0906	0.1504	0.0963	0.0552	0.0585	0.0339	0.1599	0.0913
		II	0.0258	0.0318	0.0748	0.0787	0.1125	0.0682	0.0697	0.0467	0.1928	0.1227
		III	0.0161	0.0267	0.1023	0.0889	0.0802	0.0985	0.0502	0.0643	0.1333	0.1733