

E-Bayesian Estimation of Exponential-Lomax Distribution Under Asymmetric Loss Functions

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ABSTRACT

Exponential and Lomax has wide applications in statistical practices because of their wide mathematical tracability. They are used in modelling the business failure time data, survival analysis as well as modelling the income data. Exponential distribution exhibits a constant failure rate and also has the fundamental property of being memoryless. The Lomax distribution is the special case of Pareto distribution for which the support begins at zero and also known as Pareto-II distribution. In this paper, we have estimated the unknown scale parameter θ of Exponential-Lomax Distribution using Bayesian and E-Bayesian estimation under Degroot, Al-Bayyati and Minimum-Expected loss functions by considering gamma as a conjugate prior distribution. A Monte Carlo simulation technique has been used to obtain the numerical results of Bayesian and E-Bayesian estimates along with their respective MSEs that has been obtained by using MatLab.

Keywords: Bayesian, E-Bayesian, Minimum Expected loss function, Degroot loss function, Al-Bayyati loss function, Gamma Conjugate Prior.

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1 Introduction

The distributions like Exponential and Lomax are widely used distributions for life time analysis, modelling the income data and business failure time data. Because of their great mathematical tracability, they have several important statistical properties. Exponential distribution is similar in one way to the geometric distribution and it has the key property of being memoryless. It is the only continuous probability distribution that has a constant failure rate. In addition to being used for the analysis of Poisson point processes it has found applications in various other

contexts too. The Lomax distribution is Pareto Type-II distribution and is used in business, economics, actuarial sciences etc. The support of lomax distribution begins at zero. Various extensions and generalizations have been introduced of lomax distributions. Recently, a new functional form named as extended Lomax distribution was introduced by (Lemonte, 2011). Further, a new generalization of the lomax distribution called Exponential-Lomax (ELomax) distribution has been proposed by (Bassiouny, Abdo and Shahan, 2015). In the fitting of real life data sets this distribution has been proved to be highly competitive.

The two parameter cumulative density function of Exponential-Lomax distribution for known scale parameter is given by :

$$F(x) = 1 - e^{-\theta(x+1)^{-\alpha}}, \quad x > 0, \alpha, \theta > 0.$$

The probability density function of Exponential-Lomax distribution is given by

$$f(x) = \alpha\theta \cdot (x+1)^{\alpha-1} \cdot e^{-\theta(x+1)^{-\alpha}}, \quad x > 0, \alpha, \theta > 0 \quad (1.1)$$

The corresponding reliability function of Exponential-Lomax distribution is

$$R(x) = e^{-\theta(x+1)^{-\alpha}}, \quad x > 0, \alpha, \theta > 0.$$

where, α is a shape parameter and θ is a scale parameter.

The history of exponential density function has been traced by (Balakrishnan and Sandhu, 1995) in their seminal work. (Mudholka, D.K. and Freimer, 1995) proposed the EWD as a generalization of the standard weibull distribution which is a suitable model for bus motor failure time data. (Aljouharah, 2013) estimated the parameters of Exponentiated Inverted Weibul distribution under Type-II censoring. (Din, Riad and Sayad, 2014) worked on statistical inference and prediction for the IWD based on record data. (Ahmad, 2015) discussed the Bayesian estimator of shape parameter of EIWD under different loss function namely precautionary loss function, entropy loss function and square error loss function. (Kumar, Kour and Kour, 2018) studied the MLE and UMVUE of shape parameter and probability density function of lomax distribution.

Bayesian estimation approach has received great consideration by the researchers and has been proven better than the classical estimators. A new approach of Bayesian estimation called Expected Bayesian method has been introduced by (Han, 2007) in which he obtained

the Expected Bayesian estimate of failure probability by considering quadratic loss function and discussed the properties of Expected Bayesian estimate and showed that Expected Bayesian estimate is efficient and accessible to operate. (Jaheen and Okasha, 2011) compared the Bayesian with Expected Bayesian estimator for the parameters and reliability function of the Burr-XII model for Type-II censoring by considering the squared error loss function and LINEX loss function and have shown that the performance of Expected Bayesian estimate is better. (Okasha, 2012) estimated Bayesian and Expected Bayesian method for estimating the scale parameter of the Weibull Distribution under Type-II censored samples by considering the SELF. (Azimi, Yaghomei and Fasihi, 2013) studied E-Bayesian estimation of GHL progressive Type-II censored data. (Reyad and Ahmed, 2015) developed E-Bayesian estimators of parameter of the Gumbel Type-II distribution. (Pak and Kadivar, 2016) estimated parameter for Exponential-distribution that was based on imprecise data of doubly Type-II censoring. (Patel and M.N., 2017) used the Bayesian approach to Optimal warranty length for an Exponential Life time distributed based on Type-II censoring Scheme. (Ahmad, 2018) estimated the Bayesian parameters of Maxwell-Boltzmann distribution using techniques of normal and T-K Approximation.

This paper is aimed to estimate the unknown parameters of Exponential-Lomax Distribution and has been organized in the following way. In section 2, Bayes estimator has been derived using Minimum Expected, Degroot and Al-Bayyati loss functions using gamma conjugate prior. In Section 3, Expected-Bayesian estimators has been obtained for the unknown parameter θ . Properties of Expected Bayesian estimates are discussed in Section 4. Monte Carlo simulation is described in section 5. Numerical results of Bayesian and Expected Bayesian along with their respective mean square errors has been obtained by using MatLab. In section 6, concluding remarks have been outlined. Finally, Appendix has been given in section 7.

2 Bayesian Estimation

In estimation theory, a Bayesian estimator is as an estimator which minimizes the posterior expected value of a loss function. Bayesian estimators consider the parameters as random variables instead of unknown constants that are used in classical estimation procedures. In this section, Bayesian estimators of the unknown scale parameter of EL are obtained by using gamma informative prior under three different loss functions. Let a random sample $X_1, X_2, \dots, X_n$ of size n be drawn from EL with probability density function given by eq. (1.1),

then the likelihood function is given by

$$L(x; \alpha, \theta) = (\alpha\theta)^n \prod_{i=1}^n (x_i + 1)^{-\alpha+1} \cdot e^{-\theta \sum_{i=1}^n (x_i+1)^{-\alpha}} \quad (2.1)$$

As we are using the gamma distribution as an conjugate prior distribution of θ then the probability density function of gamma prior distribution with shape parameter c and scale parameter r is

$$g(\theta|c, r) = \frac{r^c}{\Gamma(c)} \cdot \theta^{c-1} \cdot e^{-r\theta} \quad (2.2)$$

Combining (2.1) and (2.2), we get,

$$L \propto \alpha^n \theta^n \prod_{i=1}^n (x_i + 1)^{-\alpha+1} \cdot e^{-\theta \left(\sum_{i=1}^n (x_i+1)^{-\alpha} + r \right)} \cdot \frac{r^c}{\Gamma(c)} \theta^{c-1}$$

The posterior distribution is obtained by solving

$$\begin{aligned} P(\theta|\underline{x}) &= \frac{L(x; \alpha, \theta) g(\theta|c, r)}{\int_0^\infty L(x; \alpha, \theta) g(\theta|c, r) d\theta} \\ &= \frac{e^{-\theta \left(\sum_{i=1}^n (x_i+1)^{-\alpha} + r \right)} \cdot \theta^{n+c-1}}{\int_0^\infty e^{-\theta \left(\sum_{i=1}^n (x_i+1)^{-\alpha} + r \right)} \cdot \theta^{n+c-1} d\theta} \end{aligned}$$

On simplifications, we get,

$$P(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^n (x_i + 1)^{-\alpha} + r \right)^{n+c}}{\Gamma(n+c)} \cdot \theta^{n+c-1} \cdot e^{-\theta \left(\sum_{i=1}^n (x_i+1)^{-\alpha} + r \right)} \quad (2.3)$$

which is the required expression for posterior distribution.

2.1 Bayesian estimate of θ under Degroot Loss Function using Gamma Prior

(Degroot, 1970) discussed the bayes estimates under these loss functions. Asymmetric loss function is one such example defined for the positive values of the parameter.

If $\hat{\theta}$ is an estimate of θ , then the Degroot loss function is given by

$$L(\hat{\theta}) = \frac{\theta - \hat{\theta}}{\hat{\theta}} \quad (2.4)$$

And, the Bayes estimate using Degroot loss function is given by

$$\hat{\theta}_{DG} = \frac{E(\theta^2|\underline{x})}{E(\theta|\underline{x})} \quad (2.5)$$

We have,

$$E(\theta|\underline{x}) = \int_0^\infty \theta P(\theta|\underline{x}) d\theta = \frac{(n+c)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \quad (2.6)$$

and,

$$E(\theta^2|\underline{x}) = \int_0^\infty \theta^2 P(\theta|\underline{x}) d\theta = \frac{(n+c+1)(n+c)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)^2} \quad (2.7)$$

Therefore, using (2.6) and (2.7) in (2.5), we get,

$$\hat{\theta}_{DG} = \frac{(n+c+1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \quad (2.8)$$

which is the final expression for bayes estimator of θ under Degroot loss function.

2.2 Bayesian Estimate of θ under Al-Bayyati Loss Function using Gamma Prior

(Bayyati, 2002)) introduced a new loss function defined by :

$$L(\theta, \hat{\theta}) = \theta^{c_1} (\hat{\theta} - \theta)^2; \quad c_1 \in R \quad (2.9)$$

The Al-Bayyati loss function introduces the additional parameter c_1 , which helps in investigating a flatter loss function and it specifically generalizes the squared error loss function (SELF). c_1 can also be useful in order of weighting of the quadratic component.

The risk function of $\hat{\theta}$ under Al-Bayyati loss function is defined by

$$\begin{aligned} R_{AB}(\hat{\theta}_{AB}) &= E(L(\theta, \hat{\theta})) = \int_0^\infty \theta^{c_1} (\hat{\theta} - \theta)^2 \cdot P(\theta|\underline{x}) d\theta \\ &= \hat{\theta}^2 \frac{\Gamma(n+c+c_1)}{\Gamma(n+c) \left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)^{c_1}} + \frac{\Gamma(n+c+c_1+2)}{\Gamma(n+c) \left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)^{c_1+2}} \end{aligned} \quad (2.10)$$

$$-2\hat{\theta} \frac{\Gamma(n+c+c_1+1)}{\Gamma(n+c) \left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)^{c_1+1}} \quad (2.11)$$

Differentiating (2.10) w.r.t. $\hat{\theta}$ and equating to zero, we get,

$$\hat{\theta}_{AB} = \frac{(n+c+c_1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \quad (2.12)$$

which is the required Bayesian estimator under Al-Bayyati loss function.

2.3 Bayesian estimate of θ under Minimum Expected Loss function using Gamma Prior

(Tummala and Sathe, 1978) developed minimum expected loss function which can be obtained by

$$L(\theta, \hat{\theta}) = \frac{(\hat{\theta} - \theta)^2}{\theta^2} \quad (2.13)$$

where, $\hat{\theta}$ is an estimator of θ , we have,

$$\begin{aligned} R_{ME}(\hat{\theta}_{ME}) &= E(L(\theta, \hat{\theta})) = \int_0^\infty \frac{(\hat{\theta} - \theta)^2}{\theta^2} \cdot P(\theta|\underline{x}) d\theta \\ &= \hat{\theta}^2 \cdot \frac{\left(\sum_{i=1}^n (x_i + 1)^\alpha + r\right)^2}{(n + c - 1)(n + c - 2)} + 1 - 2\hat{\theta} \cdot \frac{\left(\sum_{i=1}^n (x_i + 1)^\alpha + r\right)}{(n + c - 1)} \end{aligned} \quad (2.14)$$

Differentiating (2.13) w.r.t. $\hat{\theta}$ and equating the obtained differential equal to zero simplifications, we get the following expression :

$$\hat{\theta}_{ME} = \frac{(n + c - 2)}{\left(\sum_{i=1}^n (x_i + 1)^\alpha + r\right)} \quad (2.15)$$

which is the required expression for Bayesian estimator under Minimum Expected loss function.

3 E-Bayesian Estimation

Han (1997) discussed that the prior parameters c and r should be selected in such a manner that it guarantee the prior $g(\theta|c, r)$ in (2.2) to be a decreasing function of θ . Here,

$$g(\theta|c, r) = \frac{\theta^c}{\Gamma(c)} e^{-\theta r} \cdot \theta^{c-1} \quad (3.1)$$

The derivative of (3.1) with respect to θ is

$$\frac{d}{d\theta} g(\theta|c, r) = \frac{r^c}{\Gamma(c)} \theta^{c-2} \cdot e^{-\theta r} [(c - 1) - r\theta]$$

Therefore, for hyperparameters $0 < c < 1$ and $r > 0$, the prior distribution $g(\theta|c, r)$ is a decreasing function of θ .

The E-Bayesian estimate of θ implies the expectation of the Bayesian estimate of the Bayesian estimate of θ is given by

$$\hat{\theta}_{EB} = E(\theta|\underline{x}) = \int \int_Q \hat{\theta}_{BE} \cdot \pi(\theta, c, r) dr dc;$$

Here, Q is the domain of the hyperparameters c and r for which the prior density function is decreasing in θ and $\hat{\theta}$ is the Bayesian estimate of θ under the different loss functions (Han, 2007). To obtain the Expected Bayesian estimates of θ , we need to choose prior distribution of hyperparameters c and r . These distributions are used to study the impact of different prior distributions on E-Bayesian estimation of θ . The following distributions of c and r are given by

$$\pi_1(\theta, c, r) = \frac{2(s-r)}{s^2} ; 0 < c < 1, 0 < r < s \quad (3.2)$$

$$\pi_2(\theta, c, r) = \frac{1}{s} ; 0 < c < 1, 0 < r < s \quad (3.3)$$

$$\pi_3(\theta, c, r) = \frac{2r}{s^2} ; 0 < c < 1, 0 < r < s \quad (3.4)$$

Theorem 3.1. : (E-Bayesian Estimation of θ under Degroot Loss Function)

Let $X_1, X_2, \dots, X_n$ be independently and identically distributed random variables following the Exponential-Lomax distribution. The Expected Bayesian estimates of scale parameter θ under the Degroot loss function using prior $\pi_1(\theta, c, r)$, denoted by $\hat{\theta}_{EBDG_1}$ can be obtained by using (2.8) and (3.2), we get,

$$\begin{aligned} \hat{\theta}_{EBDG_1} &= \int_0^1 \int_0^s \frac{(n+c+1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{2(s-r)}{s^2} dr dc \\ &= \frac{(2n+3)}{s^2} \cdot \left[\left(\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right) \left(s + \sum_{i=1}^n (x_i+1)^\alpha \right) - s \right] \end{aligned} \quad (3.5)$$

E-Bayesian estimates of θ under the Degroot loss function based on prior $\pi_2(\theta, c, r)$, is denoted by $\hat{\theta}_{EBDG_2}$ can be obtained by using (2.8) and (3.3).

$$\begin{aligned} \hat{\theta}_{EBDG_2} &= \int_0^1 \int_0^s \frac{(n+c+1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{1}{s} dr dc \\ &= \frac{1}{s} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right] \cdot \left(\frac{2n+3}{2} \right) \end{aligned} \quad (3.6)$$

Next, the E-Bayesian estimates of θ under the Degroot loss function using prior $\pi_3(\theta, c, r)$, is denoted by $\hat{\theta}_{EBDG_3}$ can be obtained by using (2.8) and (3.4).

$$\begin{aligned} \hat{\theta}_{EBDG_3} &= \int_0^1 \int_0^s \frac{(n+c+1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{2r}{s^2} dr dc \\ &= \left(\frac{2n+3}{s^2} \right) \cdot \left[s - \sum_{i=1}^n (x_i+1)^\alpha \cdot \left(\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right) \right] \end{aligned} \quad (3.7)$$

Theorem 3.2. : (E-Bayesian Estimation of θ under Al-Bayyati Loss Function)

Suppose $X_1, X_2, \dots, X_n$ be independently and identically distributed random variables following Exponential-Lomax distribution. Then, the Expected Bayesian estimates of parameter θ

under the Al-Bayyati loss function based on prior $\pi_1(\theta, c, r)$, is denoted by $\hat{\theta}_{EBAB_1}$ can be estimated by using (2.12) and (3.2), we get,

$$\begin{aligned}\hat{\theta}_{EBAB_1} &= \int_0^1 \int_0^s \frac{(n+c+c_1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{2(s-r)}{s^2} dr dc \\ &= \frac{(2n+1+2c_1)}{s^2} \cdot \left[\left(\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right) \left(s + \sum_{i=1}^n (x_i+1)^\alpha \right) - s \right] \quad (3.8)\end{aligned}$$

E-Bayesian estimation of θ under the Al-Bayyati loss function using prior $\pi_2(\theta, c, r)$, is denoted by $\hat{\theta}_{EBAB_2}$, obtained by using (2.12) and (3.3), we get,

$$\begin{aligned}\hat{\theta}_{EBAB_2} &= \int_0^1 \int_0^s \frac{(n+c+c_1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{1}{s} dr dc \\ &= \frac{(2n+1+2c_1)}{2s} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right] \quad (3.9)\end{aligned}$$

E-Bayesian estimation of θ under the Al-Bayyati loss function using prior $\pi_3(\theta, c, r)$, is denoted by $\hat{\theta}_{EBAB_3}$, obtained by using (2.12) and (3.4), we get,

$$\begin{aligned}\hat{\theta}_{EBAB_3} &= \int_0^1 \int_0^s \frac{(n+c+c_1)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{2r}{s^2} dr dc \\ &= \left(\frac{2n+1+2c_1}{s^2} \right) \cdot \left[s - \sum_{i=1}^n (x_i+1)^\alpha \cdot \log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right] \quad (3.10)\end{aligned}$$

Theorem 3.3. : (E-Bayesian Estimation of θ under Minimum Expected Loss Function)

Let $X_1, X_2, \dots, X_n$ be independently and identically distributed random variables following the Exponential-Lomax distribution. Therefore, the Expected Bayesian estimates of parameter θ relative to Minimum Expected loss function based on prior $\pi_1(\theta, c, r)$, denoted by $\hat{\theta}_{EBME_1}$ can be obtained by using (2.17) and (3.2), we get,

$$\begin{aligned}\hat{\theta}_{EBME_1} &= \int_0^1 \int_0^s \frac{(n+c-2)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{2(s-r)}{s^2} dr dc \\ &= \frac{(2n-3)}{s^2} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \left(s + \sum_{i=1}^n (x_i+1)^\alpha \right) - s \right] \quad (3.11)\end{aligned}$$

E-Bayesian estimation of θ under Minimum Expected loss function which is using prior $\pi_2(\theta, c, r)$, denoted by $\hat{\theta}_{EBME_2}$ and is obtained by using (2.17) and (3.3), we get,

$$\hat{\theta}_{EBME_2} = \int_0^1 \int_0^s \frac{(n+c-2)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{1}{s} dr dc = \log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{2n-3}{2s} \right) \quad (3.12)$$

Next, the E-Bayesian estimation of θ relative to Minimum Expected loss function using prior

$\pi_3(\theta, c, r)$, denoted by $\hat{\theta}_{EBME_3}$ and can be obtained by using (2.17) and (3.4), we get,

$$\begin{aligned}\hat{\theta}_{EBME_3} &= \int_0^1 \int_0^s \frac{(n+c-2)}{\left(\sum_{i=1}^n (x_i+1)^\alpha + r\right)} \cdot \frac{2r}{s^2} dr dc \\ &= \left(\frac{2n-3}{s^2}\right) \cdot \left[s - \sum_{i=1}^n (x_i+1)^\alpha \cdot \log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha}\right)\right]\end{aligned}\quad (3.13)$$

4 Properties of E-Bayesian Estimators

To study the relations among different E-Bayesian estimators obtained under Degroot, Al-Bayyati and Minimum-Expected loss functions, that is, $\hat{\theta}_{EBDG_i}$, $\hat{\theta}_{EBAB_i}$ and $\hat{\theta}_{EBME_i}$ ($i=1, 2, 3$), we have derived the relations and are presented in the following theorems :

Theorem 4.1. *Expected Bayesian estimates obtained using Degroot loss function follow the following results:*

- I) $\hat{\theta}_{EBDG_3} < \hat{\theta}_{EBDG_2} < \hat{\theta}_{EBDG_1}$
- II) $\lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBDG_1} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBDG_2} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBDG_3}$

Theorem 4.2. *Expected Bayesian estimates obtained using Al-Bayyati loss function follow the following results:*

- I) $\hat{\theta}_{EBAB_3} < \hat{\theta}_{EBAB_2} < \hat{\theta}_{EBAB_1}$
- II) $\lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBAB_1} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBAB_2} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBAB_3}$

Theorem 4.3. *Expected Bayesian estimates obtained using Minimum Expected loss function follow the following results:*

- I) $\hat{\theta}_{EBME_3} < \hat{\theta}_{EBME_2} < \hat{\theta}_{EBME_1}$
- II) $\lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBME_1} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBME_2} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBME_3}$

Note : See Appendix for the proof of above theorems.

5 Monte-Carlo Simulation Study and Comparisons

In this section, a Monte-Carlo simulation study was conducted for comparisons of Bayes and E-Bayes estimator in the following steps :

1. For given values of parameter s , we generate sample of r from uniform distribution $U(0, s)$.
2. Now, for given values of c and r , we generate θ from the gamma prior distribution given in (3.1).
3. For known values of θ samples of sizes 10, 20, 50, 100, 200 and 400 are generated from posterior distribution of *Exponential – Lomax*(α, θ) given in (2.3).
4. Next, using the samples obtained in previous step, we compute the estimates $\hat{\theta}_{DG}$, $\hat{\theta}_{EBDG_1}$, $\hat{\theta}_{EBDG_2}$ and $\hat{\theta}_{EBDG_3}$ of θ from (2.13), (3.7), (3.8) and (3.11) respectively based on Degroot

Loss function,.

5. Based on Al-Bayyati Loss function, we compute the estimates $\hat{\theta}_{AB}$, $\hat{\theta}_{EBAB1}$, $\hat{\theta}_{EBAB2}$ and $\hat{\theta}_{EBAB3}$ of θ from (2.17), (3.12), (3.16) and (3.11) respectively.

6. Based on Minimum Expected Loss function, we compute the estimates $\hat{\theta}_{ME}$, $\hat{\theta}_{EBME1}$, $\hat{\theta}_{EBME2}$ and $\hat{\theta}_{EBME3}$ of θ from (2.22), (3.18), (3.19) and (3.22) respectively.

7. We then compute the mean values and mean square error (MSE) for the estimates.

8. The computational results are listed in Tables (1-7).

Table 1: Bayesian and E-Bayesian estimates of parameter $\theta = 1.5$ under Degroot Loss function.

n	$\alpha \downarrow$	$\theta \downarrow$	Bayesian	E-Bayesian1	E-Bayesian2	E-Bayesian3
10	1.0	1.5	1.1059	1.1100	1.1069	1.1028
				1.1139	1.1102	1.1074
				1.1187	1.1165	1.1121
	0.5	1.0	1.1119	1.1099	1.1063	1.1030
				1.1090	1.1176	1.1193
				1.1236	1.1210	1.1198
	0.5	0.5	1.0978	1.1056	1.1035	1.1018
				1.1159	1.1176	1.1193
				1.1136	1.1140	1.1097
20	1.0	1.5	1.1139	1.1287	1.1248	1.1229
				1.1259	1.1226	1.1184
				1.1215	1.1180	1.1157
	0.5	1.0	1.1120	1.1269	1.1240	1.1219
				1.1229	1.1216	1.1183
				1.1215	1.1180	1.1167
	0.5	0.5	1.1107	1.1238	1.1219	1.1202
				1.1214	1.1206	1.1173
				1.1200	1.1188	1.1164
50	1.0	1.5	1.1489	1.1677	1.1642	1.1599
				1.1594	1.1556	1.1513
				1.1540	1.1518	1.1484
	0.5	1.0	1.1256	1.1448	1.1421	1.1380
				1.1374	1.1339	1.1283
				1.1341	1.1311	1.1287
	0.5	0.5	1.1058	1.1358	1.1339	1.1309
				1.1236	1.1220	1.1193
				1.1212	1.1188	1.1164

n	$\alpha \downarrow$	$\theta \downarrow$	Bayesian	E-Bayesian1	E-Bayesian2	E-Bayesian3
100	1.0	1.5	1.2069	1.2195	1.2163	1.2132
				1.2155	1.2116	1.2073
				1.2120	1.2108	1.2074
	0.5	1.0	1.2041	1.2160	1.2138	1.2115
				1.2132	1.2116	1.2093
				1.2110	1.2088	1.2064
	0.5	0.5	1.2013	1.2139	1.2118	1.2080
				1.2113	1.2102	1.2063
				1.2089	1.2068	1.2044
200	1.0	1.5	1.3589	1.3692	1.3651	1.3624
				1.3523	1.3488	1.3443
				1.3412	1.3378	1.3347
	0.5	1.0	1.3693	1.3787	1.3766	1.3743
				1.3674	1.3646	1.3613
				1.3642	1.3578	1.3544
	0.5	0.5	1.3288	1.3389	1.3366	1.3349
				1.3362	1.3341	1.3283
				1.3310	1.3288	1.3264
400	1.0	1.5	1.4889	1.4953	1.4951	1.4948
				1.4861	1.4832	1.4786
				1.4821	1.4808	1.4784
	0.5	1.0	1.4322	1.4845	1.4835	1.4822
				1.4766	1.4726	1.4703
				1.4725	1.4708	1.4674
	0.5	0.5	1.4226	1.4468	1.4453	1.4449
				1.4382	1.4336	1.4313
				1.4351	1.4328	1.4288

Table 2: Bayesian and E-Bayesian estimates of parameter $\theta = 1.5$ under Al-Bayyati Loss function.

n	$\alpha \downarrow$	$\theta \downarrow$	Bayesian	E-Bayesian1	E-Bayesian2	E-Bayesian3
10	1.0	1.5	1.1174	1.1286	1.1250	1.1221
				1.1234	1.1206	1.1183
				1.1210	1.1198	1.1164
	0.5	1.0	1.1138	1.1268	1.1237	1.1203
				1.1244	1.1226	1.1203
				1.1216	1.1208	1.1184
	0.5	0.5	1.1066	1.1168	1.1136	1.1078
				1.1134	1.1116	1.1083
				1.1107	1.1078	1.1052

n	$\alpha \downarrow$	$\theta \downarrow$	Bayesian	E-Bayesian1	E-Bayesian2	E-Bayesian3
20	1.0	1.5	1.1368	1.1484	1.1468	1.1445
				1.1458	1.1436	1.1403
				1.1430	1.1418	1.1384
	0.5	1.0	1.1357	1.1471	1.1456	1.1434
				1.1445	1.1426	1.1383
				1.1420	1.1408	1.1364
	0.5	0.5	1.1320	1.1448	1.1429	1.1411
				1.1421	1.1406	1.1373
				1.1407	1.1388	1.1364
50	1.0	1.5	1.1655	1.1785	1.1764	1.1737
				1.1752	1.1736	1.1713
				1.1687	1.1658	1.1624
	0.5	1.0	1.1632	1.1788	1.1761	1.1730
				1.1758	1.1739	1.1719
				1.1740	1.1708	1.1684
	0.5	0.5	1.1610	1.1764	1.1734	1.1673
				1.1732	1.1700	1.1687
				1.1687	1.1667	1.1634
100	1.0	1.5	1.2149	1.2353	1.2310	1.2296
				1.2310	1.2286	1.2243
				1.2289	1.2268	1.2234
	0.5	1.0	1.2073	1.2322	1.2290	1.2261
				1.2282	1.2256	1.2233
				1.2260	1.2238	1.2184
	0.5	0.5	1.2013	1.2298	1.2239	1.2199
				1.2235	1.2176	1.2153
				1.2210	1.2180	1.2164
200	1.0	1.5	1.3988	1.4197	1.4139	1.4087
				1.4094	1.4056	1.4033
				1.4050	1.4038	1.3994
	0.5	1.0	1.3774	1.3981	1.3964	1.3912
				1.3875	1.3856	1.3833
				1.3841	1.3818	1.3784
	0.5	0.5	1.3578	1.3699	1.3660	1.3627
				1.3594	1.3556	1.3503
				1.3540	1.3518	1.3484
400	1.0	1.5	1.4932	1.5253	1.5225	1.5218
				1.5194	1.5156	1.5123
				1.5169	1.5138	1.5104
	0.5	1.0	1.4748	1.5142	1.5129	1.5112
				1.5084	1.5056	1.5013
				1.5040	1.5018	1.4984
	0.5	0.5	1.4594	1.4668	1.4622	1.4611
				1.4574	1.4546	1.4523
				1.4540	1.4518	1.4484

Table 3: Bayesian and E-Bayesian estimates of parameter $\theta = 1.5$ under Minimum Expected Loss function.

n	$\alpha \downarrow$	$\theta \downarrow$	Bayesian	E-Bayesian1	E-Bayesian2	E-Bayesian3
10	1.0	1.5	1.1289	1.1377	1.1342	1.1299
				1.1294	1.1256	1.1203
				1.1240	1.1218	1.1184
	0.5	1.0	1.1256	1.1348	1.1321	1.1280
				1.1274	1.1239	1.1183
				1.1221	1.1201	1.1167
	0.5	0.5	1.1121	1.1158	1.1139	1.1119
				1.1136	1.1120	1.1093
				1.1112	1.1108	1.1064
20	1.0	1.5	1.1410	1.1549	1.1520	1.1178
				1.1510	1.1476	1.1453
				1.1490	1.1478	1.1452
	0.5	1.0	1.1357	1.1491	1.1456	1.1434
				1.1475	1.1456	1.1433
				1.1450	1.1428	1.1413
	0.5	0.5	1.1357	1.1469	1.1447	1.1421
				1.1441	1.1426	1.1413
				1.1429	1.1418	1.1384
50	1.0	1.5	1.1715	1.1885	1.1864	1.1837
				1.1852	1.1836	1.1813
				1.1787	1.1758	1.1724
	0.5	1.0	1.1732	1.1868	1.1831	1.1890
				1.1828	1.1779	1.1749
				1.1770	1.1758	1.1724
	0.5	0.5	1.1710	1.1784	1.1754	1.1723
				1.1762	1.1740	1.1727
				1.1747	1.1722	1.1684
100	1.0	1.5	1.2219	1.2453	1.2410	1.2396
				1.2410	1.2386	1.2343
				1.2389	1.2368	1.2334
	0.5	1.0	1.2163	1.2352	1.2310	1.2271
				1.2322	1.2286	1.2263
				1.2290	1.2268	1.2234
	0.5	0.5	1.2113	1.2398	1.2339	1.2289
				1.2335	1.2276	1.2253
				1.2310	1.2281	1.2264
200	1.0	1.5	1.4088	1.4297	1.4239	1.4187
				1.4194	1.4156	1.4133
				1.4150	1.4138	1.4094
	0.5	1.0	1.3884	1.4081	1.4054	1.3972
				1.3980	1.3956	1.3933
				1.3941	1.3928	1.3894
	0.5	0.5	1.3688	1.3892	1.3862	1.3837
				1.3694	1.3656	1.3623
				1.3640	1.3618	1.3584

n	$\alpha \downarrow$	$\theta \downarrow$	Bayesian	E-Bayesian1	E-Bayesian2	E-Bayesian3
400	1.0	1.5	1.5072	1.5372	1.5345	1.5328
				1.5294	1.5256	1.5223
				1.5269	1.5238	1.5214
	0.5	1.0	1.4848	1.5242	1.5229	1.5212
				1.5174	1.5177	1.5143
				1.5140	1.5118	1.5084
	0.5	0.5	1.4694	1.4768	1.4732	1.4711
				1.4674	1.4646	1.4623
				1.4640	1.4618	1.4588

Table 4: MSE Values of Bayesian and E-Bayesian estimators for parameter $\theta = 1.5$ with respect to different sample sizes n .

For $\alpha = 0.5$, $\theta = 0.5$.

n	10	20	50	100	200	400
$\hat{\theta}_{DG}$	0.7972	0.7025	0.5972	0.2977	0.0066	0.0009
$\hat{\theta}_{AB}$	0.5692	0.4261	0.1997	0.0081	0.0019	0.0008
$\hat{\theta}_{ME}$	0.4332	0.4291	0.3099	0.2331	0.0118	0.0016
$\hat{\theta}_{EBDG1}$	0.6573	0.6028	0.5027	0.2655	0.0059	0.0008
$\hat{\theta}_{EBDG2}$	0.5122	0.4664	0.3745	0.2189	0.0041	0.0005
$\hat{\theta}_{EBDG3}$	0.4189	0.3815	0.3279	0.1918	0.0036	0.0002
$\hat{\theta}_{EBAB1}$	0.4933	0.3999	0.1739	0.0328	0.0017	0.0007
$\hat{\theta}_{EBAB2}$	0.3725	0.3689	0.1556	0.0258	0.0013	0.00005
$\hat{\theta}_{EBAB3}$	0.3051	0.2924	0.1248	0.0661	0.0009	0.0002
$\hat{\theta}_{EBME1}$	0.3844	0.3555	0.2667	0.2098	0.0100	0.00014
$\hat{\theta}_{EBME2}$	0.3666	0.2899	0.2109	0.1756	0.0094	0.00011
$\hat{\theta}_{EBME3}$	0.2897	0.2654	0.1778	0.1431	0.0081	0.00008

For $\alpha = 0.5$, $\theta = 1.0$

n	10	20	50	100	200	400
$\hat{\theta}_{DG}$	0.6548	0.6428	0.5768	0.3549	0.0999	0.0045
$\hat{\theta}_{AB}$	0.4844	0.4317	0.1717	0.1025	0.0085	0.0006
$\hat{\theta}_{ME}$	0.5548	0.3955	0.2778	0.1947	0.0248	0.0018
$\hat{\theta}_{EBDG1}$	0.5637	0.5523	0.5519	0.3215	0.0815	0.0038
$\hat{\theta}_{EBDG2}$	0.4799	0.4530	0.5228	0.3058	0.0710	0.0027
$\hat{\theta}_{EBDG3}$	0.4565	0.4322	0.4400	0.2836	0.0523	0.0015
$\hat{\theta}_{EBAB1}$	0.3745	0.3822	0.1512	0.1001	0.0072	0.0005
$\hat{\theta}_{EBAB2}$	0.3444	0.3222	0.1321	0.0998	0.0065	0.0004
$\hat{\theta}_{EBAB3}$	0.2979	0.2969	0.1763	0.0846	0.0051	0.0001
$\hat{\theta}_{EBME1}$	0.5247	0.3756	0.2546	0.1846	0.0199	0.00015
$\hat{\theta}_{EBME2}$	0.4777	0.3110	0.2112	0.1625	0.0162	0.00011
$\hat{\theta}_{EBME3}$	0.3619	0.2665	0.1823	0.1478	0.0110	0.00006

For $\alpha = 1.0, \theta = 1.5$

n	10	20	50	100	200	400
$\hat{\theta}_{DG}$	0.6534	0.6286	0.4766	0.3992	0.1006	0.0672
$\hat{\theta}_{AB}$	0.5752	0.4470	0.2045	0.1257	0.0125	0.0027
$\hat{\theta}_{ME}$	0.4598	0.4339	0.3238	0.2667	0.0999	0.0098
$\hat{\theta}_{EBDG1}$	0.6182	0.5774	0.4455	0.3215	0.0898	0.0501
$\hat{\theta}_{EBDG2}$	0.5736	0.5217	0.4187	0.2989	0.0656	0.0366
$\hat{\theta}_{EBDG3}$	0.4409	0.4288	0.3662	0.2284	0.0422	0.0124
$\hat{\theta}_{EBAB1}$	0.5274	0.4078	0.1881	0.1058	0.0108	0.0022
$\hat{\theta}_{EBAB2}$	0.4833	0.3671	0.1680	0.0925	0.0099	0.0017
$\hat{\theta}_{EBAB3}$	0.3511	0.3201	0.1322	0.0749	0.0056	0.0010
$\hat{\theta}_{EBME1}$	0.4164	0.4195	0.2914	0.2478	0.0774	0.00076
$\hat{\theta}_{EBME2}$	0.3722	0.3642	0.2331	0.2108	0.0532	0.00055
$\hat{\theta}_{EBME3}$	0.3347	0.3325	0.1785	0.1466	0.0328	0.00032

Table 5: Values of upper limit, lower limit and length of Interval for Bayesian Credible Interval under Degroot Loss function with respect to different sample sizes n

. For $\alpha = 1.0$ and $\theta = 1.5$

n	Lower limit	Upper limit	Length of Interval
10	1.0210	1.2382	0.2172
20	1.0525	1.2380	0.1855
50	1.2788	1.4187	0.1399
100	1.3156	1.4255	0.1099
200	1.3955	1.4554	0.0599
400	1.4599	1.4950	0.0351

For $\alpha = 0.5$ and $\theta = 1.0$

n	Lower limit	Upper limit	Length of Interval
10	1.0189	1.2422	0.2233
20	1.0498	1.2499	0.2001
50	1.2655	1.4233	0.1578
100	1.3079	1.4388	0.1309
200	1.3877	1.4611	0.0734
400	1.4519	1.4996	0.0477

For $\alpha = 0.5$ and $\theta = 0.5$

n	Lower limit	Upper limit	Length of Interval
10	1.0178	1.2497	0.2319
20	1.0412	1.2536	0.2124
50	1.2605	1.4289	0.1684
100	1.2955	1.4433	0.1478
200	1.3776	1.4682	0.0906
400	1.4474	1.5014	0.0540

Table 6: Values of upper limit, lower limit and length of Interval for Bayesian Credible Interval for Al-Bayyati New Loss function with respect to different sample sizes n .

For $\alpha = 1.0$ and $\theta = 1.5$

n	Lower limit	Upper limit	Length of Interval
10	1.0390	1.2272	0.1882
20	1.1466	1.3087	0.1621
50	1.3249	1.4473	0.1224
100	1.3888	1.5044	0.1056
200	1.4654	1.5160	0.0506
400	1.4647	1.5322	0.0457

For $\alpha = 0.5$ and $\theta = 1.0$

n	Lower limit	Upper limit	Length of Interval
10	1.0490	1.2355	0.1865
20	1.1299	1.3028	0.1729
50	1.3129	1.4588	0.1459
100	1.3769	1.4768	0.1056
200	1.4582	1.5190	0.0608
400	1.4789	1.5188	0.0399

For $\alpha = 0.5$ and $\theta = 0.5$

n	Lower limit	Upper limit	Length of Interval
10	1.0298	1.2276	0.2276
20	1.1166	1.3188	0.2022
50	1.2464	1.3841	0.1377
100	1.3499	1.4487	0.0988
200	1.4029	1.4487	0.0458
400	1.4333	1.4742	0.0287

Table 7: Values of upper limit, lower limit and length of Interval for Bayesian Credible Interval for Minimum-Expected Loss function with respect to different sample sizes n

For $\alpha = 1.0$ and $\theta = 1.5$

n	Lower limit	Upper limit	Length of Interval
10	1.0699	1.2687	0.1988
20	1.0949	1.2812	0.1863
50	1.3456	1.4881	0.1425
100	1.3917	1.4899	0.0982
200	1.4768	1.5200	0.0432
400	1.4998	1.5256	0.0258

For $\alpha = 0.5$ and $\theta = 1.0$

n	Lower limit	Upper limit	Length of Interval
10	1.0989	1.3097	0.2108
20	1.1227	1.3204	0.1977
50	1.3377	1.4836	0.1459
100	1.3559	1.4561	0.1002
200	1.4455	1.5076	0.0621
400	1.4667	1.4903	0.0236

For $\alpha = 0.5$ and $\theta = 0.5$

n	Lower limit	Upper limit	Length of Interval
10	1.0376	1.2498	0.2122
20	1.0919	1.2917	0.1998
50	1.3234	1.4662	0.1428
100	1.3282	1.4418	0.0936
200	1.4259	1.4778	0.0519
400	1.4369	1.4725	0.0356

Plots between MSE values of MLE, Bayes and E-Bayes estimates under Degroot, Al-Bayyati and Minimum-Expected Loss functions for different values of α and θ .

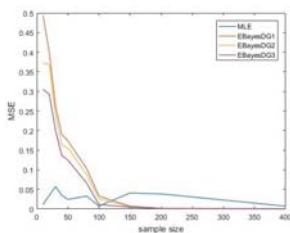


Figure 1: MSE values of MLE and Bayes estimators under Degroot, Al-Bayyati and Minimum-Expected Loss functions for $\alpha = \theta = 0.5$.

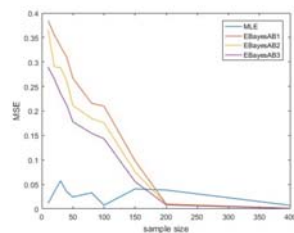


Figure 2: MSE values of MLE and Bayes estimators under Degroot Loss functions for $\alpha = \theta = 0.5$.

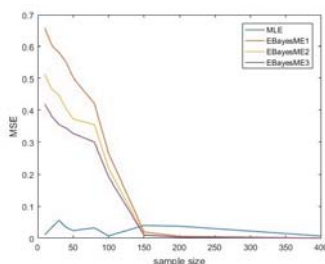


Figure 3: MSE values of MLE and Bayes estimators under Al-Bayyati Loss functions for $\alpha = \theta = 0.5$.

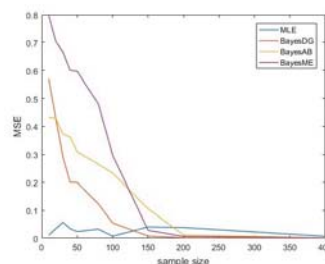


Figure 4: MSE values of MLE and Bayes estimators under Minimum-Expected Loss functions for $\alpha = \theta = 0.5$.

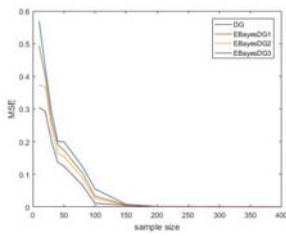


Figure 5: MSE values of Bayes and E-Bayes estimators under Degroot Loss functions for $\alpha = \theta = 0.5$.

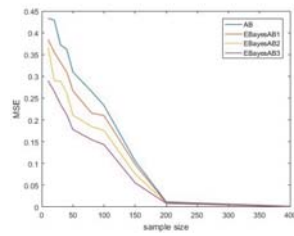


Figure 6: MSE values of Bayes and E-Bayes estimators under Al-Bayyati Loss functions for $\alpha = \theta = 0.5$.

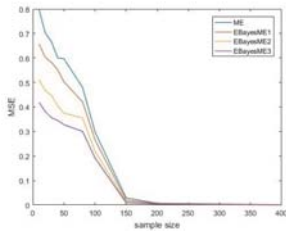


Figure 7: MSE values of Bayes and E-Bayes estimators under Minimum-Expected Loss functions for $\alpha = \theta = 0.5$.

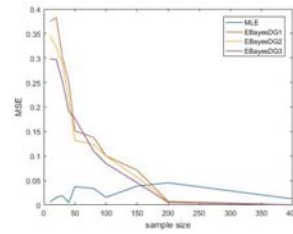


Figure 8: MSE values of Bayes and E-Bayes estimators under Degroot Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

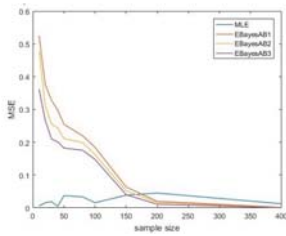


Figure 9: MSE values of Bayes and E-Bayes estimators under Al-Bayyati Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

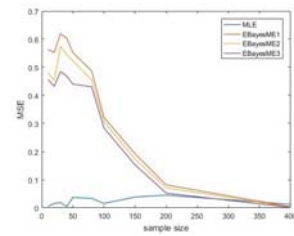


Figure 10: MSE values of Bayes and E-Bayes estimators under Minimum-Expected Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

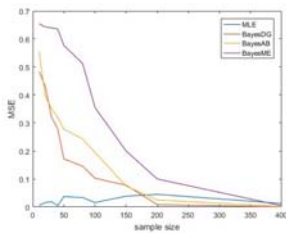


Figure 11: MSE values of MLE and Bayes estimators under Degroot, Al-Bayyati and Minimum-Expected Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

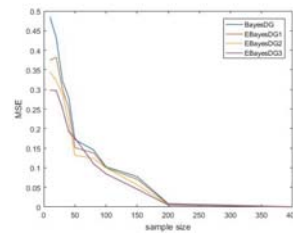


Figure 12: MSE values of BayesDG and E-Bayes estimators under Degroot Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

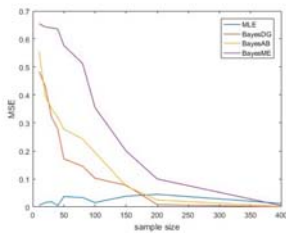


Figure 13: MSE values of BayesAB and Bayes estimators under AI-Bayyati Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

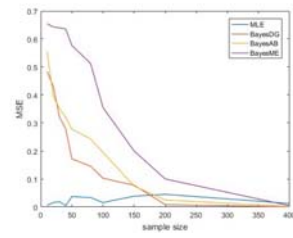


Figure 14: MSE values of BayesME and Bayes estimators under Minimum-Expected Loss functions for $\alpha = 0.5$ and $\theta = 1.0$.

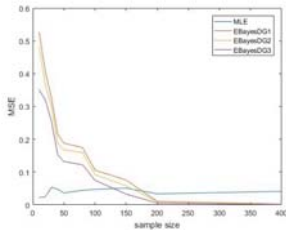


Figure 15: MSE values of MLE and Bayes estimators under Degroot Loss functions for $\alpha = 1.0$ and $\theta = 1.5$.

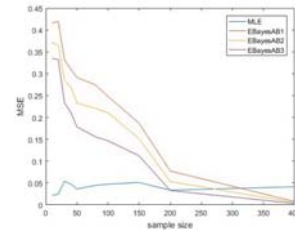


Figure 16: MSE values of MLE and Bayes estimators under AI-Bayyati Loss functions for $\alpha = 1.0$ and $\theta = 1.5$.

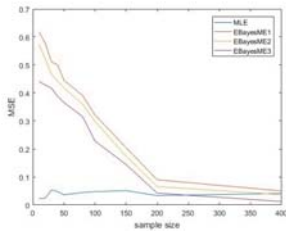


Figure 17: MSE values of MLE and Bayes estimators under Minimum-Expected Loss functions for $\alpha = 1.0$ and $\theta = 1.5$.

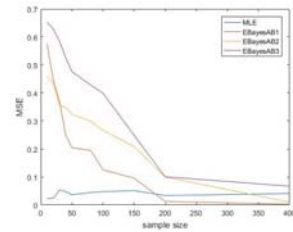


Figure 18: MSE values of MLE and Bayes estimators under Degroot, AI-Bayyati and Minimum-Expected Loss functions for $\alpha = 1.0$ and $\theta = 1.5$.

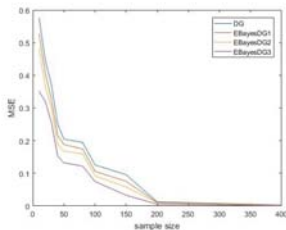


Figure 19: MSE values of BayesDG and Bayes estimators under Degroot Loss functions for $\alpha = 1.0$ and $\theta = 1.5$.

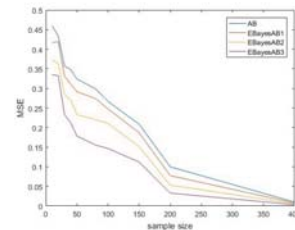


Figure 20: MSE values of BayesAB and Bayes estimators under AI-Bayyati Loss functions for $\alpha = 1.0$ and $\theta = 1.5$.

6 Conclusion

In this paper, we have estimated Bayesian and E-bayesian estimates of parameter θ for Exponential-Lomax distribution. Based on the results shown in Tables 1,2 and 3 one can conclude that E-bayesian provides better estimates than bayesian estimates. Generally, the MSE of the E-bayesian estimates are the smallest MSE as the as compared with the bayesian estimates. From tables 4, 5 and 6, we see that MSE of E-Bayesian estimates of θ have the smallest values as compared with the MSE of bayesian estimates. Also, we can conclude that E-bayesian estimates under Al-Bayyati loss function have smallest MSE as compared with the E-bayesian estimates under Degroot loss function and minimum expected loss function. It is immediate to note that MSE of bayesian and E-bayesian estimates decrease as n increases.

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7 Appendix

Proof 1:

I. From (3.5) and (3.8), we have,

$$\begin{aligned}\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} &= \frac{(2n+3)}{s^2} \cdot \left[\left(\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \right) \left(s + \sum_{i=1}^n (x_i + 1)^\alpha \right) - s \right] \\ &\quad - \frac{1}{s} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \right] \cdot \left(\frac{2n+3}{2} \right) \\ &= \left(\frac{2n+3}{s} \right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i + 1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right]\end{aligned}\quad (7.1)$$

Next, from (3.8) and (3.11), we get,

$$\begin{aligned}\hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3} &= \frac{1}{s} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \right] \cdot \left(\frac{2n+3}{2} \right) \\ &\quad - \left(\frac{2n+3}{s^2} \right) \cdot \left[s - \sum_{i=1}^n (x_i + 1)^\alpha \cdot \left(\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \right) \right] \\ &= \left(\frac{2n+3}{s} \right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i + 1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right]\end{aligned}\quad (7.2)$$

Now from (4.1) and (4.2) we get,

$$\begin{aligned}\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} &= \hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3} \\ &= \left(\frac{2n+3}{s}\right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i + 1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right]\end{aligned}\quad (7.3)$$

For $-1 < y < 1$, we have,

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \quad (7.4)$$

Assuming, $y = \frac{s}{\sum_{i=1}^n (x_i + 1)^\alpha}$; when $0 < \frac{s}{\sum_{i=1}^n (x_i + 1)^\alpha} < 1$, after simplifications we get the following expression,

$$\begin{aligned}\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2} &= \hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3} \\ &= \left(\frac{2n+3}{s}\right) \cdot \left[\frac{s^2}{12 \left(\sum_{i=1}^n (x_i + 1)^\alpha \right)^2} + \frac{s^2}{6 \left(\sum_{i=1}^n (x_i + 1)^\alpha \right)^3} - \dots \right] \\ &> 0\end{aligned}\quad (7.5)$$

Therefore,

$$\hat{\theta}_{EBDG_3} < \hat{\theta}_{EBDG_2} < \hat{\theta}_{EBDG_1} \quad (7.6)$$

II. From (4.3), we have,

$$\begin{aligned}&\lim_{\sum_{i=1}^n (x_i + 1)^\alpha \rightarrow \infty} (\hat{\theta}_{EBDG_1} - \hat{\theta}_{EBDG_2}) = \lim_{\sum_{i=1}^n (x_i + 1)^\alpha \rightarrow \infty} (\hat{\theta}_{EBDG_2} - \hat{\theta}_{EBDG_3}) \\ &= \lim_{\sum_{i=1}^n (x_i + 1)^\alpha \rightarrow \infty} \left(\frac{2n+3}{s}\right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i + 1)^\alpha}{\sum_{i=1}^n (x_i + 1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i + 1)^\alpha}{s} + \frac{1}{2} - 1 \right) \right] = 0.\end{aligned}$$

That is,

$$\lim_{\sum_{i=1}^n (x_i + 1)^\alpha \rightarrow \infty} \hat{\theta}_{EBDG_1} = \lim_{\sum_{i=1}^n (x_i + 1)^\alpha \rightarrow \infty} \hat{\theta}_{EBDG_2} = \lim_{\sum_{i=1}^n (x_i + 1)^\alpha \rightarrow \infty} \hat{\theta}_{EBDG_3} \quad (7.7)$$

Thus, the proof is complete.

Proof 2 :

I. From (3.12) and (3.13), we have,

$$\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2}$$

$$\begin{aligned}
 &= \frac{(2n+1+2c_1)}{s^2} \cdot \left[\left(\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right) \left(s + \sum_{i=1}^n (x_i+1)^\alpha \right) - s \right] \\
 &- \left(\frac{(2n+1+2c_1)}{2s} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right] \right) \\
 &= \left(\frac{(2n+1+2c_1)}{s} \right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i+1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right] \quad (7.8)
 \end{aligned}$$

From (3.13) and (3.16), we get,

$$\begin{aligned}
 &\hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3} \\
 &= \frac{(2n+1+2c_1)}{2s} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right] \\
 &- \left[\left(\frac{(2n+1+2c_1)}{s^2} \right) \cdot \left(s - \sum_{i=1}^n (x_i+1)^\alpha \cdot \log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right) \right] \\
 &= \left(\frac{(2n+1+2c_1)}{s} \right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i+1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right] \quad (7.9)
 \end{aligned}$$

Therefore, from (4.7) and (4.8), we get,

$$\begin{aligned}
 &\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} \\
 &= \left(\frac{(2n+1+2c_1)}{s} \right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i+1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right] \quad (7.10)
 \end{aligned}$$

For $-1 < y < 1$, we have,

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \quad (7.11)$$

Assuming, $y = \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha}$; when $0 < \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} < 1$, after simplifications we get the following expression,

$$\begin{aligned}
 &\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} = \hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3} \\
 &= \left(\frac{(2n+1+2c_1)}{s} \right) \cdot \left[\frac{s^2}{12 \left(\sum_{i=1}^n (x_i+1)^\alpha \right)^2} + \frac{s^2}{6 \left(\sum_{i=1}^n (x_i+1)^\alpha \right)^3} - \dots \right] \\
 &> 0 \quad (7.12)
 \end{aligned}$$

Therefore,

$$\hat{\theta}_{EBAB_3} < \hat{\theta}_{EBAB_2} < \hat{\theta}_{EBAB_1} \quad (7.13)$$

II. From (4.10), we have,

$$\begin{aligned} \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} (\hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2}) &= \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} (\hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3}) \\ &= \left(\frac{2n+1+2c_1}{s} \right) \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{\sum_{i=1}^n (x_i+1)^\alpha}{s} + \frac{1}{2} \right) - 1 \right] = 0. \end{aligned} \quad (7.14)$$

That is,

$$\lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBAB_1} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBAB_2} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBAB_3} \quad (7.15)$$

which completes the proof. **Proof 3 :**

I. From (3.18) and (3.19), we have,

$$\begin{aligned} &\hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2} \\ &= \frac{(2n-3)}{s^2} \cdot \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(s + \sum_{i=1}^n (x_i+1)^\alpha \right) - s \right] \\ &\quad - \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{2n-3}{2s} \right) \right] \end{aligned} \quad (7.16)$$

$$= \frac{(2n-3)}{s} \left[\log \left(1 + \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{s + 2\sum_{i=1}^n (x_i+1)^\alpha}{2s} \right) - 1 \right] \quad (7.17)$$

From (3.19) and (3.21), we have,

$$\begin{aligned} &\hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3} \\ &= \left[\log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{2n-3}{2s} \right) \right] \\ &\quad - \left[\left(\frac{2n-3}{s^2} \right) \cdot \left(s - \sum_{i=1}^n (x_i+1)^\alpha \cdot \log \left(\frac{s + \sum_{i=1}^n (x_i+1)^\alpha}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \right) \right] \\ &= \frac{(2n-3)}{s} \left[\log \left(1 + \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{s + 2\sum_{i=1}^n (x_i+1)^\alpha}{2s} \right) - 1 \right] \end{aligned} \quad (7.18)$$

From (4.17) and (4.18), we get,

$$\hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2} = \hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3}$$

$$= \frac{(2n-3)}{s} \left[\log \left(1 + \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{s + 2\sum_{i=1}^n (x_i+1)^\alpha}{2s} \right) - 1 \right] \quad (7.19)$$

For $-1 < y < 1$, we have,

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \dots \quad (7.20)$$

Assuming, $y = \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha}$; when $0 < \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} < 1$, after simplifications we get the following expression, therefore,

$$\begin{aligned} \hat{\theta}_{EBAB_1} - \hat{\theta}_{EBAB_2} &= \hat{\theta}_{EBAB_2} - \hat{\theta}_{EBAB_3} \\ &= \left(\frac{2n-3}{s} \right) \cdot \left[\frac{s^2}{12 \left(\sum_{i=1}^n (x_i+1)^\alpha \right)^2} + \frac{s^2}{6 \left(\sum_{i=1}^n (x_i+1)^\alpha \right)^3} - \dots \right] \\ &> 0 \end{aligned} \quad (7.21)$$

Therefore,

$$\hat{\theta}_{EBEM_3} < \hat{\theta}_{EBEM_2} < \hat{\theta}_{EBEM_1} \quad (7.22)$$

II. From (4.19), we have,

$$\begin{aligned} \hat{\theta}_{EBME_1} - \hat{\theta}_{EBME_2} &= \hat{\theta}_{EBME_2} - \hat{\theta}_{EBME_3} \\ &= \frac{(2n-3)}{s} \left[\log \left(1 + \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{s + 2\sum_{i=1}^n (x_i+1)^\alpha}{2s} \right) - 1 \right] \end{aligned}$$

Now,

$$\begin{aligned} \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} (\hat{\theta}_{EBEM_1} - \hat{\theta}_{EBEM_2}) &= \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} (\hat{\theta}_{EBEM_2} - \hat{\theta}_{EBEM_3}) \\ &= \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \frac{(2n-3)}{s} \left[\log \left(1 + \frac{s}{\sum_{i=1}^n (x_i+1)^\alpha} \right) \cdot \left(\frac{s + 2\sum_{i=1}^n (x_i+1)^\alpha}{2s} \right) - 1 \right] \\ &= 0 \end{aligned} \quad (7.23)$$

Therefore, we get,

$$\lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBEM_1} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBEM_2} = \lim_{\sum_{i=1}^n (x_i+1)^\alpha \rightarrow \infty} \hat{\theta}_{EBEM_3} \quad (7.24)$$

Thus, the proof is complete.

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