

Strong uniform consistency rates of the local linear estimation of the conditional hazard estimator for functional data

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ABSTRACT

This paper focuses on the conditional hazard function a scalar response variable given a random variable taking values in a semi-metric space. Employing the local linear estimator of the conditional density and cumulative distribution function. We study the uniform almost complete consistencies with convergence rate under some generale conditions for proposed estimator.

Keywords: Functional data, nonparametric local linear estimation, conditional hazard estimator, uniform almost complete convergence.

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1 Introduction

Nonparametric estimation of the conditional hazard function is a substantial issue in data analysis, it's instrument in survival analysis and many other fields (econometrics, medicine, seismology).

In recent decades, the functional estimate has been a great interest and attracted attention by statistical scientist, for their relevance and use in all areas and to help with rapid progress in most fields, the nonparametric estimation of the hazard function play an important role in the statistics, this function was first introduced by (Watson and Leadbetter 1964), after that coming authors they worked in this domain, some works and their authors are mentioned for example, asymptotic normality of the hazard rate function with dependence (Roussa 1989), uniform almost complete convergence properties and asymptotic normality of an estimate of the maximum hazard function (Quintela Rio 2006), almost complete convergence studied by (Massim

and Mechab 2016), the mean squared error was established by (Merouan, Mechab and Massim 2018), treated the spatial case: the almost complete convergence with rate (Laksaci and Mechab 2010) .

However in the finite dimensional data case, local linear estimation method has different benefit, it well known that the kernel method is inferior to the local linear method consequent to the restrictions such as large bias, non adaptation and edge effects, we can takes a few works. In fact (Fan 1992, Fan and Yao 2003), around nonparametric estimation on the local linear for functional data have also been investigated recently, (Baillo and Grame 2009), they first put a local linear estimation of the regression operator when the explanatory variable takes values in a Hilbert space, (Demongeot et al. 2011), established both pointwise almost complete consistencies with convergence rate of the conditional density estimation.

In this work we introduce a new nonparametric estimation of the conditional hazard function for functional data, this estimation is based on the local linear estimation, the goal of this paper is to study under some ganarale condtions uniform almost complete convergence with rate.

The article is organized as follows: section 2 introduces the locale linear estimation of conditional hazard function for functional data. The assumptions and some remarks are given in section 3. In section 4 we show the main asymptotic results, we collect all the proofs of this paper in the last section.

2 The model

We consider $\{(X_i, Y_i), 1 \leq i \leq n\}$ be n random variables independent identically distribution (i.i.d), as (X, Y) which is a random pair valued in $\mathcal{F} \times \mathbb{R}$, where $\mathcal{F}$ is a semi-metric space equipped with a semi-metric d . In the following x will be a fixed point in $\mathcal{F}$, N_x will denote a fixed neighborhood of x and $S_{\mathbb{R}}$ will be fixed compact subset of $\mathbb{R}$.

Our goal in this paper to estimate the conditional hazard function h^x using n independent observations (X_i, Y_i) draw from a random variables with the same distribution with (X, Y) where the regular version F^x of the conditional distribution function of Y given $X = x$ exists for any $x \in \mathcal{F}$. Moreover we suppose that F^x has a continuous density f^x with respect to (w.r.t) Lebesgue's measure over $\mathbb{R}$. We adopt here functional local linear estimator of the conditional distribution function $\hat{F}^x$ is estimated by $\hat{a}$ where the couple $(\hat{a}, \hat{b})$ is obtained by minimizing the following quantity:

$$(\hat{a}, \hat{b}) = \arg \min_{(a,b) \in \mathbb{R}^2} \sum_{i=1}^n (H(h_H^{-1}(y - Y_i)) - a - b\beta(X_i, x))^2 K(h_H^{-1}\delta(x, X_i)) \quad (2.1)$$

where $\beta(.,.)$ and $\delta(.,.)$ are known bi-functional operators defined from $\mathcal{F}^2$ into $\mathbb{R}$, such that:

$$\forall \xi \in \mathcal{F}, \beta(\xi, \xi) = 0 \text{ and } d(.,.) = |\delta(.,.)|$$

With the function K is kernel, H is a distribution function and $h_K = h_{K,n}$ (respectively, $h_H = h_{H,n}$) is chosen as a sequence of positive real numbers. It's clearly the estimator $\hat{a}$, given by

(2.1), can be explicitly defined by:

$$\hat{F}^x(y) = \frac{\sum_{1 \leq i, j \leq n} W_{ij}(x) H(h_H^{-1}(y - Y_j))}{\sum_{1 \leq i, j \leq n} W_{ij}(x)}, \quad \forall y \in \mathbb{R} \quad (2.2)$$

where

$$W_{ij}(x) = \beta(X_i, x)(\beta(X_i, x) - \beta(X_j, x))K(h_K^{-1}\delta(x, X_i))K(h_K^{-1}\delta(x, X_j))$$

with the convention $0/0 = 0$.

Furthermore, the density function estimator $\hat{f}^x$ of f^x can be obtained from the distribution function and we defined by

$$\hat{f}^x(y) = \frac{\sum_{1 \leq i, j \leq n} W_{ij}(x) H'(h_H^{-1}(y - Y_j))}{h_H \sum_{1 \leq i, j \leq n} W_{ij}(x)}, \quad \forall y \in \mathbb{R}$$

Where H' is the derivative of H .

Finally, we estimate the conditional hazard function h^x by:

$$\hat{h}^x(y) = \frac{\hat{f}^x(y)}{1 - \hat{F}^x(y)}, \quad \forall y \in \mathbb{R}.$$

3 Main results

In this part, we investigate the uniform almost complete convergence of the estimator $\hat{h}^x$ on some subset $S_{\mathcal{F}}$ of $\mathcal{F}$, such that:

$$S_{\mathcal{F}} \subset \bigcup_{k=1}^{d_n} B(x_k, r_n)$$

where $x_k \in \mathcal{F}$ and r_n (resp, d_n) is a sequence of positive real numbers. All along the paper, x will denote a fixed point in $\mathcal{F}$ and $\phi_x(r_1, r_2) = \mathbb{P}(r_2 \leq \delta(X, x) \leq r_1)$.

Furthermore, we need the following assumptions:

(H1) There exists a differentiable function $\phi(\cdot)$, such that:

$$\forall x \in S_{\mathcal{F}}, \quad 0 < C\phi(h) \leq \phi_x(h) \leq C'\phi(h) < \infty \text{ and } \exists \eta_0 > 0, \forall \eta < \eta_0, \phi'(\eta) < C,$$

where C and C' are strictly positive constants and where ϕ' denotes the first derivative of ϕ .

(H2) The conditional distribution function F^x (resp. density function f^x) is such that: there exist $b_1 > 0$ and $b_2 > 0$,

$$\forall (y_1, y_2) \in S_{\mathbb{R}} \times S_{\mathbb{R}}, \forall (x_1, x_2) \in S_{\mathcal{F}} \times S_{\mathcal{F}}$$

$$|F^{x_1(j)}(y_1) - F^{x_2(j)}(y_2)| \leq C \left(\delta|(x_1, x_2)|^{b_1} + |y_1 - y_2|^{b_2} \right), j = 0, 1$$

where C is a positive constant depending on x .

(H3) The function $\delta(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are such that:

$$\forall z \in \mathcal{F}, |\delta(x, z)| = d(x, z) \text{ and } C_1 |\delta(x, z)| \leq |\beta(x, z)| \leq C_1 |\delta(x, z)|,$$

where $C_1 > 0, C_2 > 0$,

and, for some strictly positive constant C' , the following Lipschitz's condition:

$$\forall (x_1, x_2) \in \mathcal{S}_{\mathcal{F}} \times \mathcal{S}_{\mathcal{F}} : |\beta(x_1, z) - \beta(x_2, z)| \leq C' |\delta(x_1, x_2)|.$$

(H4) K is a positive, differentiable function with support $[-1, 1]$ and, for some strictly positive constant C , the following Lipschitz's condition:

$$|K(x) - K(y)| \leq C ||x| - |y||.$$

(H5) The kernel H is a differentiable function which has a bounded first derivative such that:

$$\int |t|^{b_2} H^{(j)}(t) dt < \infty \text{ and } \int H^{(j)^2}(t) dt < \infty. \text{ for } j \in \{0, 1\}$$

$$\forall (y_1, y_2) \in \mathbb{R}^2, |H'(y_1) - H'(y_2)| \leq C |y_1 - y_2|,$$

H' is a bounded function.

(H6) The bandwidth h_K satisfies:

$$\exists n_0 \in \mathbb{N}, \forall n > n_0, -\frac{1}{\Phi_x(h_K)} \int_{-1}^1 \phi_x(z h_K, h_K) \frac{d}{dz} (z^2 K(z)) dz > C_3 > 0$$

and

$$h_K \int_{B(x, h_K)} \beta(u, x) dP(u) = o \left(\int_{B(x, h_K)} \beta^2(u, x) dP(u) \right),$$

where $B(x, r) = \{z \in \mathcal{F} / |\delta(z, x)| \leq r\}$ and $dP(x)$ is the cumulative distribution of X .

(H7) for $r_n = O\left(\frac{\ln n}{n}\right)$, and for some $\gamma \in (0, 1)$, for n large enough the sequence d_n satisfies:

$$\frac{(\ln n)^2}{n^{1-\gamma} h_H^j \phi(h_k)} < \ln d_n < \frac{n^{1-\gamma} h_H^j \phi(h_k)}{\ln n} \text{ for } j \in \{0, 2\}$$

and

$$\sum_{n=1}^{\infty} n^{\frac{(3\gamma+1)}{2}} d_n^{1-\beta} < \infty, \text{ for some } \beta > 1,$$

conditions (H1), (H3) and (H6) are the same as those used in Barrientos-Marin et al. (2010) and Demongeot et al. (2013). While assumption (H2) is a regularity condition which characterizes the functional space of our model and it is needed to evaluate the bias term in the asymptotic results of this paper. Remember that this hypothesis is similar to the assumption (H2) in Ferraty et al. (2006). (H5) is technical condition.

Theorem 3.1. *Under assumptions (H1)-(H6), we obtain:*

$$\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |\hat{h}^x(y) - h^x(y)| = O(h_K^{b_1} + h_H^{b_2}) + O_{a.co} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right).$$

Proof of Theorem 3.1. The proof of theorem 3.1 is a direct consequence of the following decomposition:

$$\hat{h}^x(y) - h^x(y) = \frac{1}{1 - \hat{F}^x(y)} [\hat{f}^x(y) - f^x(y)] + \frac{h^x(y)}{1 - \hat{F}^x(y)} [\hat{F}^x(y) - F^x(y)]. \quad (3.1)$$

And it's based on following results

$$\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |\hat{F}^x(y) - F^x(y)| = O(h_K^{b_1} + h_H^{b_2}) + O_{a.co} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right)$$

$$\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |\hat{f}^x(y) - f^x(y)| = O(h_K^{b_1} + h_H^{b_2}) + O_{a.co} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right)$$

$$\exists \eta > 0 \text{ such that } \sum_{n=1}^{\infty} \mathbb{P} \left\{ \inf_{x \in S_{\mathcal{F}}} |1 - \hat{F}^x(y)| < \eta \right\} < \infty.$$

Whose demonstrations are based, respectively, on the decompositions

$$\begin{aligned} \hat{F}^x(y) - F^x(y) &= \frac{1}{\hat{F}_D^x} \left\{ \left(\hat{F}_N^x(y) - \mathbb{E}[\hat{F}_N^x(y)] \right) - \left(F^x(y) - \mathbb{E}[\hat{F}_N^x(y)] \right) \right\} \\ &+ \frac{F^x(y)}{\hat{F}_D^x} (1 - \hat{F}_D^x) \end{aligned}$$

and

$$\begin{aligned} \hat{f}^x(y) - f^x(y) &= \frac{1}{\hat{F}_D^x} \left\{ \left(\hat{f}_N^x(y) - \mathbb{E}[\hat{f}_N^x(y)] \right) - \left(f^x(y) - \mathbb{E}[\hat{f}_N^x(y)] \right) \right\} \\ &+ \frac{f^x(y)}{\hat{F}_D^x} (1 - \hat{F}_D^x) \end{aligned}$$

Where

$$\hat{F}_D^x = \frac{1}{n(n-1)\mathbb{E}[W_{12}(x)]} \sum_{i \neq j} W_{ij}(x),$$

and

$$\hat{F}_N^x(y) = \frac{1}{n(n-1)\mathbb{E}[W_{12}(x)]} \sum_{i \neq j} W_{ij}(x) H(h_H^{-1}(y - Y_j)),$$

also

$$\hat{f}_N^x(y) = \frac{1}{n(n-1)h_H\mathbb{E}[W_{12}(x)]} \sum_{i \neq j} W_{ij}(x) H'(h_H^{-1}(y - Y_j)).$$

Lemma 3.2. *under assumptions (H1), (H3)-(H6), we obtain that:*

$$\sup_{x \in S_F} |\hat{F}_D^x - 1| = O_{a.co} \left(\sqrt{\frac{\ln d_n}{n\phi(h_K)}} \right).$$

Corollary 3.3. *Under the assumptions of Lemma 3.2, we have that:*

$$\sum_{n=1}^{\infty} \mathbb{P} \left(\inf_{x \in S_F} \hat{F}_D^x < \frac{1}{2} \right) < \infty.$$

Lemma 3.4. *Under assumptions (H1), (H2) and (H5), we obtain that:*

$$\sup_{x \in S_F} \sup_{y \in S_{\mathbb{R}}} |f^x(y) - \mathbb{E}[\hat{f}_N^x(y)]| = O(h_K^{b_1}) + O(h_H^{b_2}).$$

Lemma 3.5. *Under assumptions (H1), (H2) and (H5), we obtain that:*

$$\sup_{x \in S_F} \sup_{y \in S_{\mathbb{R}}} |F^x(y) - \mathbb{E}[\hat{F}_N^x(y)]| = O(h_K^{b_1}) + O(h_H^{b_2}).$$

Lemma 3.6. *Under the assumptions of Theorem 3.1, we get:*

$$\sup_{x \in S_F} \sup_{y \in S_{\mathbb{R}}} |\hat{f}_N^x(y) - \mathbb{E}[\hat{f}_N^x(y)]| = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma}\phi(h_K)}} \right).$$

Lemma 3.7. *Under the assumptions of Theorem 3.1, we get:*

$$\sup_{x \in S_F} \sup_{y \in S_{\mathbb{R}}} |\hat{F}_N^x(y) - \mathbb{E}[\hat{F}_N^x(y)]| = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma}\phi(h_K)}} \right).$$

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4 Appendix

In what follows, we define the quantities, for any $x \in \mathcal{F}$, and for all $i = 1, \dots, n$

$$K_i(x) = K(h^{-1}\delta(x, X_i)), \beta_i(x) = \beta(X_i, x) \text{ and } H_i(y) = H(h_H^{-1}(y - Y_i)).$$

Proof of Lemma 3.2.

The proof of this lemma is based on the same decomposition kind, we have:

$$\hat{F}_D^x = \underbrace{\frac{n^2 h_K^2 \phi_x^2(h_K)}{n(n-1)\mathbb{E}}}_{E_1} \left[\underbrace{\left(\frac{1}{n} \sum_{j=1}^n \frac{K_j(x)}{\phi_x(h_K)} \right)}_{D_2(x)} \underbrace{\left(\frac{1}{n} \sum_{i=1}^n \frac{K_i(x) \beta_i^2(x)}{h_K^2 \phi_x(h_K)} \right)}_{D_4(x)} - \underbrace{\left(\frac{1}{n} \sum_{j=1}^n \frac{K_j(x) \beta_j(x)}{h_K \phi_x(h_K)} \right)}_{D_3(x)} \underbrace{\left(\frac{1}{n} \sum_{i=1}^n \frac{K_i(x) \beta_i(x)}{h_K \phi_x(h_K)} \right)}_{D_3(x)} \right].$$

Thus, all it remains to show are the following uniform convergences:

$$\sup_{x \in S_F} |D_k(x) - \mathbb{E}[D_k(x)]| = O \left(\sqrt{\frac{\ln d_n}{n\phi(h_K)}} \right). \text{ a.co. for } k = 2, 3, 4. \quad (4.1)$$

and

$$\sup_{x \in S_{\mathcal{F}}} |\mathbb{E}[D_2(x)]\mathbb{E}[D_4(x)] - \mathbb{E}[D_2(x)D_4(x)] - \text{Var}[D_3(x)]| = o\left(\sqrt{\frac{\ln d_n}{n\phi(h_K)}}\right). \text{ a.co}$$

and, also that, uniformly on $x \in S_{\mathcal{F}}$:

$$E_1 = O(1) \text{ and } |\mathbb{E}[D_k(x)]| = O(1), \text{ for } k = 2, 3, 4.$$

Clearly, the last two equations are direct consequences of assumption (H1), moreover indeed, by noting: $j(x) = \arg \min_{j \in \{1, 2, \dots, d_n\}} |\delta(x, x_k)|$, we consider the following decomposition:

$$\begin{aligned} \sup_{x \in S_{\mathcal{F}}} |D_k(x) - \mathbb{E}[D_k(x)]| &\leq \underbrace{\sup_{x \in S_{\mathcal{F}}} |D_k(x) - D_k(x_{j(x)})|}_{N_1^k} + \underbrace{\sup_{x \in S_{\mathcal{F}}} |D_k(x_{j(x)}) - \mathbb{E}[D_k(x_{j(x)})]|}_{N_2^k} \\ &\quad + \underbrace{\sup_{x \in S_{\mathcal{F}}} |\mathbb{E}[D_k(x_{j(x)})] - \mathbb{E}[D_k(x)]|}_{N_3^k}. \end{aligned}$$

Firstly, we consider the terms N_1^k and N_3^k have almost the same treatment. let us analyse the first term N_1^k for $k = 2, 3, 4$. Since K is supported within $[-1, 1]$, we can write for all $k = 2, 3, 4$ that:

$$\begin{aligned} N_1^k &\leq \frac{1}{nh_K^{k-2}\phi_x(h_K)} \sup_{x \in S_{\mathcal{F}}} \sum_{i=1}^n |K_i(x)\beta_i^{k-2}(x)1_{B(x, h_K)}(X_i) - K_i(x_{j(x)})\beta_i^{k-2}(x_{j(x)})1_{B(x_{j(x)}, h_K)}(X_i)|, \\ &\leq \frac{C(k-2)}{nh_K^{k-2}\phi_x(h_K)} \sup_{x \in S_{\mathcal{F}}} \sum_{i=1}^n |K_i(x)1_{B(x, h_K)}(X_i)|\beta_i^{k-2}(x) - \beta_i^{k-2}(x_{j(x)})1_{B(x_{j(x)}, h_K)}(X_i)| \\ &\quad + \frac{1}{nh_K^{k-2}\phi_x(h_K)} \sup_{x \in S_{\mathcal{F}}} \sum_{i=1}^n |\beta_i^{k-2}(x_{j(x)})1_{B(x_{j(x)}, h_K)}(X_i)|K_i(x)1_{B(x, h_K)}(X_i) - K_i(x_{j(x)})|. \end{aligned}$$

The Lipschitz condition on K allows us directly to write:

$$1_{B(x_{j(x)}, h_K)}(X_i)|K_i(x)1_{B(x, h_K)}(X_i) - K_i(x_{j(x)})| \leq C\epsilon 1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i) + C1_{B(x_{j(x)}, h_K) \cap \overline{B(x, h_K)}}(X_i).$$

We apply again Lipschitz condition on β to get:

$$1_{B(x, h_K)}(X_i)|\beta_i(x) - \beta_i(x_{j(x)})1_{B(x_{j(x)}, h_K)}(X_i)| \leq \epsilon 1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i) + h_K 1_{B(x, h_K) \cap \overline{B(x_{j(x)}, h_K)}}(X_i),$$

$$1_{B(x, h_K)}(X_i)|\beta_i^2(x) - \beta_i^2(x_{j(x)})1_{B(x_{j(x)}, h_K)}(X_i)| \leq \epsilon h_K 1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i) + h_K^2 1_{B(x, h_K) \cap \overline{B(x_{j(x)}, h_K)}}(X_i)$$

which implies that, for $k = 3, 4$:

$$\begin{aligned} &1_{B(x, h_K)}(X_i)|\beta_i^{k-2}(x) - \beta_i^{k-2}(x_{j(x)})1_{B(x_{j(x)}, h_K)}(X_i)| \\ &\leq \epsilon h_K^{k-3} 1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i) + h_K^{k-2} 1_{B(x, h_K) \cap \overline{B(x_{j(x)}, h_K)}}(X_i). \end{aligned}$$

Thus

$$N_1^k \leq C \sup_{x \in S_{\mathcal{F}}} \left(G_{11}^k + G_{12} + G_{13}^k + G_{14} \right),$$

where

$$\begin{aligned} N_{11}^k &= \frac{C(k-2)}{n\phi_x(h_K)} \sum_{i=1}^n 1_{B(x, h_K) \cap \overline{B(x_{j(x)}, h_K)}}(X_i), & N_{12} &= \frac{C\epsilon}{n\phi_x(h_K)} \sum_{i=1}^n 1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i), \\ N_{13}^k &= \frac{C(k-2)\epsilon}{nh_K\phi_x(h_K)} \sum_{i=1}^n 1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i), & N_{14} &= \frac{C}{n\phi_x(h_K)} \sum_{i=1}^n 1_{B(x_{j(x)}, h_K) \cap \overline{B(x, h_K)}}(X_i). \end{aligned}$$

Now, to evaluate the terms N_{11}^k , N_{12} , N_{13}^k and N_{14} , we can apply a standard inequality for sums of bounded random variables with Z_i is identified such that:

$$Z_i = \begin{cases} \frac{1}{\phi_x(h_K)} \sup_{x \in S_{\mathcal{F}}} [1_{B(x, h_K) \cap \overline{B(x_{j(x)}, h_K)}}(X_i)] & \text{for } N_{11}^k \\ \frac{\epsilon}{h_K \phi_x(h_K)} \sup_{x \in S_{\mathcal{F}}} [1_{B(x, h_K) \cap B(x_{j(x)}, h_K)}(X_i)] & \text{for } N_{12} \text{ and } N_{13}^k \\ \frac{1}{\phi_x(h_K)} \sup_{x \in S_{\mathcal{F}}} [1_{B(x_{j(x)}, h_K) \cap \overline{B(x, h_K)}}(X_i)] & \text{for } N_{14} \end{cases}$$

Clearly, under the second part of (H1), we have for the first and the last case:

$$Z_1 = O\left(\frac{1}{\phi(h_K)}\right), \quad \mathbb{E}[Z_1] = O\left(\frac{\epsilon}{\phi(h_K)}\right) \quad \text{and} \quad \text{Var}(Z_1) = O\left(\frac{\epsilon}{(\phi(h_K))^2}\right).$$

So that, we obtain:

$$N_{11}^k = O\left(\frac{\epsilon}{\phi(h_K)}\right) + O_{a.co.} \left(\sqrt{\frac{\epsilon \ln n}{n \phi(h_K)^2}} \right).$$

In the same way, assumption (H7) allows to get, for G_{12} or G_{13}^k case

$$Z_1 = O\left(\frac{\epsilon}{h_K \phi(h_K)}\right), \quad \mathbb{E}[Z_1] = O\left(\frac{\epsilon}{h_K}\right) \quad \text{and} \quad \text{Var}(Z_1) = O\left(\frac{\epsilon^2}{h_K^2 \phi(h_K)}\right)$$

which implies that:

$$N_{12}^k = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right).$$

To achieve the study of the term N_1^k , it suffices to put together all the intermediate results and use (H7) to obtain:

$$N_1^k = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right). \quad (4.2)$$

Furthermore, since:

$$N_3^k \leq \mathbb{E} \left[\sup_{x \in S_{\mathcal{F}}} |D_k(x) - D_k(x_{j(x)})| \right],$$

we obtain:

$$N_3^k = O \left(\sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right). \quad (4.3)$$

Secondly, treatment of the term G_2^k . For all $\eta > 0$, we have that:

$$\begin{aligned} \mathbb{P} \left(N_2^k > \eta \sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right) &= \mathbb{P} \left(\max_{j \in \{1, \dots, d_n\}} |D_k(x_{j(x)}) - \mathbb{E}[D_k(x_{j(x)})]| > \eta \sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right), \\ &\leq d_n \max_{j \in \{1, \dots, d_n\}} \mathbb{P} \left(|D_k(x_{j(x)}) - \mathbb{E}[D_k(x_{j(x)})]| > \eta \sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right). \end{aligned}$$

Set

$$\Delta_{ki} = \frac{1}{n h_K^{k-2} \phi(h_K)} \left(K_i(x_k) \beta_i^{k-2}(x_k) - \mathbb{E}[K_i(x_k) \beta_i^{k-2}(x_k)] \right), \text{ for } k = 2, 3, 4.$$

By using a Bernstein-type inequality, we have for all $j = 1, \dots, d_n$ and $i = 1, \dots, n$, that:

$$\mathbb{E}|\Delta_{ki}|^m = O(\phi(h_K)^{-m+1}), \text{ for } k = 2, 3, 4$$

so, we get

$$\begin{aligned} \mathbb{P} \left(|D_k(x_k) - \mathbb{E}[D_k(x_k)]| > \eta \sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right) &= \mathbb{P} \left(\frac{1}{n} \left| \sum_{i=1}^n \Delta_{ki} \right| > \eta \sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right), \\ &\leq 2 \exp\{-C\eta^2 \ln d_n\}. \end{aligned}$$

Thus, by choosing η such that $C\eta^2 = \beta$, we obtain that :

$$d_n \max_{k \in \{1, \dots, d_n\}} \mathbb{P} \left(|D_i(x_k) - \mathbb{E}[D_i(x_k)]| > \eta \sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right) \leq C' d_n^{1-\beta}. \quad (4.4)$$

Since $\sum_{i=1}^n d_n^{1-\beta} < \infty$, we obtain

$$N_2 = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n \phi(h_K)}} \right).$$

Proof of Corollary 3.3.

Clearly, we have that:

$\inf_{x \in S_{\mathcal{F}}} \widehat{F}_D(x) \leq \frac{1}{2}$ implies that exists $x \in S_{\mathcal{F}}$ such that

$$|1 - \widehat{F}_D(x)| \geq \frac{1}{2},$$

which allows to write

$$\sup_{x \in S_{\mathcal{F}}} |1 - \hat{F}_D(x)| \geq \frac{1}{2}.$$

According to the Lemma 3.2, we obtain

$$\sum_n \mathbb{P} \left(\inf_{x \in S_{\mathcal{F}}} \hat{F}_D(x) < \frac{1}{2} \right) \leq \sum_n \mathbb{P} \left(\sup_{x \in S_{\mathcal{F}}} |1 - \hat{F}_D(x)| \geq \frac{1}{2} \right) < \infty.$$

Proof of Lemma 3.4.

Since the pairs (X_i, Y_i) are identically distributed then:

$$\mathbb{E}[\hat{f}_N^x] = \mathbb{E}[W_{12}(x)[\mathbb{E}[h_H^{-1}H_1|X]]].$$

By the usual change of variables $t = \frac{y-z}{h_H}$, we obtain:

$$h_H^{-1} \mathbb{E}[H_1|X] = \int_{\mathbb{R}} H(t) f^X(y - h_H t) dt,$$

therefore

$$|\mathbb{E}[H_1|X] - f^x(y)| \leq \int_{\mathbb{R}} H(t) |f^X(y - h_H t) - f^x(y)| dt.$$

Thus, by using (H_2) and (H_4) , we obtain:

$$1_{B(x, h_K)}(X) |\mathbb{E}[H_1|X] - f^x(y)| \leq C \int_{\mathbb{R}} H(t) (h_K^{b_1} + |t|^{b_2} h_H^{b_2}) dt = O(h_K^{b_1} + h_H^{b_2}).$$

So, uniformly on x and y , we get:

$$\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |f^x(y) - \mathbb{E}[\hat{f}_N^x(y)]| = O(h_K^{b_1}) + O(h_H^{b_2}).$$

Proof of Lemma 3.6.

The proof of this lemma follows the steps as for proving Lemma 3.2, where $D_2(x)$, $D_3(x)$ and $D_4(x)$ are replaced by :

$$\begin{cases} W_2^x(y) &= \frac{1}{n} \sum_{j=1}^n \frac{K_j(x) H_j(y)}{h_H \phi_x(h_K)} \\ W_3^x(y) &= \frac{1}{n} \sum_{j=1}^n \frac{K_j(x) \beta_j(x) H_j(y)}{h_K h_H \phi_x(h_K)} \\ W_4^x(y) &= \frac{1}{n} \sum_{j=1}^n \frac{K_j(x) \beta_j^2(x) H_j(y)}{h_K^2 h_H \phi_x(h_K)} \end{cases}$$

we keep the notation used previously, namely, in the definitions of $j(x)$, t_y and l_n . The proof is based on the following decomposition:

$$\begin{aligned}
 |W_i^x(y) - \mathbb{E}[W_i^x(y)]| &\leq \underbrace{\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |V_i^x(y) - V_i^{x_{j(x)}}(y)|}_{A_1} \\
 &+ \underbrace{\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |W_i^{x_{j(x)}}(y) - W_i^{x_{j(x)}}(t_y)|}_{A_2} \\
 &+ \underbrace{\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |W_i^{x_{j(x)}}(t_y) - \mathbb{E}[W_i^{x_{j(x)}}(t_y)]|}_{A_3} \\
 &+ \underbrace{\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |\mathbb{E}[W_i^{x_{j(x)}}(t_y)] - \mathbb{E}[W_i^{x_{j(x)}}(y)]|}_{A_4} \\
 &+ \underbrace{\sup_{x \in S_{\mathcal{F}}} \sup_{y \in S_{\mathbb{R}}} |\mathbb{E}[W_i^{x_{j(x)}}(y)] - \mathbb{E}[W_i^x(y)]|}_{A_5}.
 \end{aligned}$$

Similarly to the study of the term G_1 , we obtain:

$$A_1 = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right) \text{ and } A_5 = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right), \quad (4.5)$$

for treat the term A_2 , we use the Lipschitz's condition on the kernel H and, we can write:

$$|W_i^{x_{j(x)}}(y) - W_i^{x_{j(x)}}(t_y)| \leq C \frac{1}{nh_K h_H \phi(h_K)} \sum_{i=1}^n K_i(x_{j(x)}) \beta_i^l(x_{j(x)}) |H_i(y) - H_i(t_y)| \leq \frac{l_n}{h_H^2} M_i(x_{j(x)}),$$

where $M_i(\cdot)$ for $i = 2, 3, 4$, are treated in proof of Lemma 3.2. Thus by using the facts that:

$l_n = n^{-\frac{3}{2}\gamma-1/2}$ and $(H7)$, we obtain:

$$A_2 = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right) \text{ and } A_4 = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right). \quad (4.6)$$

Finally, for the term A_3 , we have for all $\eta > 0$ that:

$$\begin{aligned}
 \mathbb{P} \left(A_3 > \eta \sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right) &= \mathbb{P} \left(\max_{j \in \{1, 2, \dots, S_n\}} \max_{k \in \{1, 2, \dots, d_n\}} |W_i^{x_{j(x)}}(t_y) - \mathbb{E}[W_i^{x_{j(x)}}(t_y)]| > \eta \sqrt{\frac{\ln d_n}{nh_H \phi(h_K)}} \right), \\
 &\leq s_n d_n \max_{j \in \{1, 2, \dots, s_n\}} \max_{k \in \{1, 2, \dots, d_n\}} \mathbb{P} \left(|W_i^{x_{j(x)}}(t_y) - \mathbb{E}[W_i^{x_{j(x)}}(t_y)]| > \eta \sqrt{\frac{\ln d_n}{nh_H \phi(h_K)}} \right).
 \end{aligned}$$

To do this last probability, we use the classical Bernstein's inequality such that, we put:

$$Z_i^l = \frac{1}{h_K^l h_H \phi(h_K)} \left(K_i(x_{j(x)}) H_i(t_y) \beta_i^l(x_{j(x)}) - \mathbb{E}[K_i(x_{j(x)}) H_i(t_y) \beta_i^l(x_{j(x)})] \right), \text{ for } l = 0, 1, 2.$$

By the assumption $(H3)$, we have that $\frac{1}{h_K^l} (K_i \beta_i^l) < C$ and since $H < 1$ then, we can write:

$$|Z_i^l| \leq \frac{C}{h_H \phi_x(h_K)} \text{ and } \mathbb{E}|Z_i^l| \leq \frac{C'}{h_H \phi_x(h_K)}.$$

Moreover, we use the classical Bernstein's inequality allows us to write for all $\eta \in (0, C'/C)$:

$$\mathbb{P} \left(\left| \sum_{i=1}^n Z_i^l \right| \geq \eta \sqrt{\frac{\ln d_n}{nh_H \phi_x(h_K)}} \right) \leq C' d_n^{-C\eta^2}$$

Therefore, the last equation allows to obtain:

$$\forall j \leq s_n, \mathbb{P} \left(|W_i^{x_{j(x)}}(t_y) - \mathbb{E}[W_i^{x_{j(x)}}(t_y)]| > \eta \sqrt{\frac{\ln d_n}{nh_H \phi(h_K)}} \right) \leq 2 \exp\{-C\eta^2 \ln d_n\},$$

since $s_n = O(n^{-\frac{3}{2}\gamma-1/2})$, $\frac{\ln d_n}{nh_H \phi(h_K)} > \frac{1}{nh_H \phi(h_K)}$ and by choosing $C\eta^2 = \beta$, one gets:

$$s_n d_n \max_{j \in \{1, 2, \dots, s_n\}} \max_{k \in \{1, 2, \dots, d_n\}} \mathbb{P} \left(|W_i^{x_{j(x)}}(t_y) - \mathbb{E}[W_i^{x_{j(x)}}(t_y)]| > \eta \sqrt{\frac{\ln d_n}{nh_H \phi(h_K)}} \right) \leq C'' s_n d_n^{1-\beta}.$$

By using the fact that $\lim_{n \rightarrow \infty} n^\gamma h_H = \infty$, and second part of condition (H7), we obtain

$$A_3 = O_{a.co.} \left(\sqrt{\frac{\ln d_n}{n^{1-\gamma} \phi(h_K)}} \right). \quad (4.7)$$

Thus, Lemma 3.6 is result can be deduced from (4.5), (4.6), (4.7).

Remark 4.1. The proof of Lemma 3.5 (resp. Lemma 3.7) follows the same eteps as for proving Lemma 3.4 (resp. Lemma 3.6) .

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