

SOME STABILITY RESULTS FOR DELAYED LOGISTIC AND MATHIEU EQUATIONS

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ABSTRACT

The stability of the delay system guaranteed by a certain condition, which is popularly known as argument principle. This paper presents a novel method of investigation of a stability of the time delayed Logistic and Mathieu equations. Method of steps is employed to analyze the stability. Also, it may be noted that the proposed method is not only easier to analyze, but also avoids tedious calculations. Moreover, there is no need of solving the complete system in order to analyze the stability which may become demanding particularly if nonlinearity existing. Some insightful illustrations are proposed to validate of the given theory

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1 Introduction

Delay differential equation DDE plays a very important role in modeling a time-delay systems over many decades. Real life phenomena enriched by the introduction of delay. Population dynamics, traffic models, epidemiology, and neuroscience are the main fields which are investigated using the concept of delay. One of the main advantage is the history over the delay interval should be provided as the initial condition. As a result of this reason and prevailing works, one may note the infinite dimensional behavior of a delay systems. Hence analytical solution is not always possible to analyze the delayed system, this leads to the adaptation of numerical solutions. It is known that stability analysis gives a mathematical tool for various systems in many subject areas especially control systems.

Enormous records can be found on this matter of stability such as Lyapunov's method, Fixed point method and Matrix measure and etc. Stability and density analysis of a mono local population growth widely studied by (Bozkurt and . Yousef, 2018). The same author investigated

global stability of an impulsive logistics equation with delay. Even (Bardi and Schiano, 1982) studied the periodic delay logistic equations. (Gao and Ding, 2007) investigated the fractional system by using Gronwall inequalities and analyzed the stability of DDE using the concept of matrix measure, meanwhile the stability of functional systems using the concept of Fixed point method is established in (Priyadharsini, 2016). Existence and stability of damped Mathieu equations widely studied by (Bauer, 2008), even (Zittouni and Amiar, 2016) studied the existence and stability of time varying delay. (Bicer, 2016) analyzed the stability of functional equation using the concept of advanced argument, also, (Wang and Wu, 2006) used the same concept in analyzing Logistic equation. Meanwhile (Huang, 2013) analyzed the time delay feedback controller for delay systems using the delay dependent design method.

The stability of fractional functional systems using the concept of Fixed point method can be seen in (Bardi and Schiano, 1982). (Priyadharsini and Govindaraj, 2018) investigated the asymptotic behavior of fractional singular systems with delays. Especially, (Bao and Wang, 2013) studied the exponential synchronization of neutral delay equations with impulses. Real life applications in the field of epidermology especially Dengue fever model has been studied by (Diaz and Bhatta, 2018) and predator prey model widely studied in (Priyadharsini, 2020). Also, it may be noted that stability analysis of time-delay systems were introduced and studied by many authors. For further details see (Bozkurt and . Yousef, 2018), (Huang, 2013), (Qian and Wong, 2010). Fundamental problems in Dynamics can be solved by the theory of delay differential equations . However, it is not always possible to get a consitant results, which meet the realities. This scenario happens only when the nonlinearity and time delays neglected. As now, it should be noted that stability of the following system (1.1) is not yet sufficiently analyzed compared to other normal systems.

$$y'(x) = ay(x) + by(x-r)y(x) + f(x, y(x)), \quad (1.1)$$

$$y(x) = \phi(x), -r \leq t \leq 0, \quad (1.2)$$

As a result, Mathieu and Logistic equations with dealy systems were analyzed. The main advantage of this proposed theory is that one may not explicitly solve the given system to analyze the stability. Many of the researcher investigated the nature and stability of Logistic equations, which is mainly applied in population studies (Bauer, 2008), (Bao and Wang, 2013), (Qian and Wong, 2010). Also, Mathieu equations have been extensively discussed and treated recently in which patterns arising from parametric square-wave forcing of an inviscid uid layer of infnite depth is basically investigated (Bicer, 2016), (Priyadharsini, 2020). Recently it has been found out that Mathieu's equation is practically applied in the following areas: the inverted pendulum, stability of a floating body, vibrations in an elliptic drum and frequency modulation. This paper is organized as follows. Section 2 includes the prevailing facts, basic definitions, theorems and lemmas . Section 3 summarizes the concept of Method of steps in order to solve delay dynamical system. In Section 4, validity of the proposed procedure is given through Logistic and Mathieu equations with delay. Also a conclusion is given in Section 5.

2 Preliminaries

This section describes the basic definition of stability and the prevailing research work on the stability of a nonlinear dynamical system which is related to this work.

Definition 2.1. Linear differential system in Cartesian form is considered

$$y'(x) = Ay(x), \quad (2.1)$$

with initial value $y(0) = y_0 = (y_{10}, y_{20}, \dots, y_{n0})^T$, where $y = (y_1, y_2, \dots, y_n)^T$, and $A \in \mathbb{R}^{n \times n}$. The autonomous system (2.1) is said to be

- (i) stable iff for any y_0 , there exists $\epsilon > 0$ such that $\|y(x)\| \leq \epsilon$ for $x \geq 0$,
- (ii) asymptotically stable iff $\lim_{t \rightarrow \infty} \|y(x)\| = 0$.

Theorem 2.1. (Priyadharsini and Govindaraj, 2018) The autonomous system (2.1) is asymptotically stable iff

$$|\arg(\text{spec}(A))| > \frac{\pi}{2} \quad (2.2)$$

Here, one may note that e^z is bounded when $|\arg(z)| > \frac{\pi}{2}$.

Lemma 2.2. (Huang, 2013)(Gronwall Inequality)

Let $g(x)$ and $\phi(x)$ are continuous in $[x_0, x_1]$, $g(x) \geq 0$, $\lambda \geq 0$ and $r \geq 0$ are two constants. If

$$\phi(x) \leq \lambda + \int_{x_0}^x [g(s)\phi(s) + r]ds, \quad \text{then} \quad (2.3)$$

$$\phi(x) \leq (\lambda + r(x_1 - x_0)) \exp\left(\int_{x_0}^x g(s)ds\right), \quad x_0 \leq x \leq x_1. \quad (2.4)$$

3 Main Result: Mathematical Theorem

Consider the general nonlinear differential equations with delay in the Cartesian form

$$y'(x) = Ay(x) + By(x-r)y(x) + f(x, y(x)),$$

with the associated initial state $y(x) = \phi(x)$, $-r \leq x \leq 0$, and r is a pure time delay with $r > 0$. Another form of initial function is $y(x_0 + \theta) = \phi(\theta)$, $-r \leq \theta \leq 0$, $\phi(\theta) \in C[-r, 0]$, where x_0 is the initial time and $C[-r, 0]$ is a Banach Space of continuous function.

The solution representation of a given system (1.1) can be found by using the method of steps (Priyadharsini, 2016). Let us consider first the case $r \in \mathbb{N}$. In this case, since $x - r \in [-1, 0]$, then $y(x - r) = \phi(x)$, and therefore equation(1.1) becomes

$$y'(x) = Ay(x) + h_1(x), \quad 0 < x \leq 1, \quad (3.1)$$

where $h_1(x) = by(x-r)y(x) + f(x, y(x))$. Since f is a continuous function, h_1 is also continuous and this equation is solvable. Its general solution is

$$y_1 = \int_0^t e^{A(x-\tau)} (B\phi(\tau-r)y(\tau) + f(\tau, y(\tau)))d\tau + e^{Ax}y_0, \quad (3.2)$$

where y_0 is a constant. It may be noted that taking (1.2) into consideration, we must have $y_0 = \phi(0)$. Hence there exists a unique solution of (1.1)-(1.2) in the interval $[0, r]$. Similarly we can proceed for the interval $[0, 2r]$. Then (1.1) may be rewritten as

$$y'(x) = Ay(x) + f(x, y(x)) + h_2(x), 0 < x \leq 2, \quad (3.3)$$

where

$$h_2(x) = \begin{cases} \phi(x-r), & 0 < x \leq r, \\ y_1(x-r), & r < x \leq 2r, \end{cases} \quad (3.4)$$

the solution in this interval exists, is unique, if $g_2(x)$ is continuous in $[0, 2r]$,

$$y_2 = \int_0^x e^{A(x-\tau)} (B\phi(\tau-r)(h_2(\tau)y(\tau) + f(\tau, y(\tau)))) d\tau + e^{Ax} c_2, \quad (3.5)$$

where c_2 is a constant. The following theorem can be concluded as follows,

Theorem 3.1. *Let $y_0(x) = \phi(x)$ and let*

$$h_k(x) = \begin{cases} y_0(x-r), & 0 < x \leq r \\ y_1(x-r), & r < x \leq 2r \\ \vdots \\ y_{(k-1)}(x-r), & (k-1)r < x \leq kr \end{cases} \quad (3.6)$$

is continuous. Here k be the greatest positive integer. Then the initial value problem (1.1)-(1.2) has a unique solution on the interval $[0, kr]$, which can be represented by $y(x) = y_i(x)$, if $(i-1)r \leq x \leq ir$, where

$$y_i(x) = \int_0^t e^{A(x-\tau)} (B\phi(\tau-r)(g_i(\tau)y(\tau) + f(\tau, y(\tau)))) d\tau + e^{Ax} c_i, \quad x \in [(i-1)r, ir]$$

and c_i is a constant, $i = 1, \dots, k$.

Method of steps is used to analyze the above theorem (Priyadharsini, 2016).

Lemma 3.2. (Bozkurt and . Yousef, 2018) *If all the eigenvalues of A satisfy*

$$|\arg(\lambda(A))| > \frac{\pi}{2}, \quad (3.7)$$

then there exists a constant $C > 0$ such that,

$$\int_0^t \|e^{A\theta}\| d\theta \leq C. \quad (3.8)$$

This lemma is proved in (Priyadharsini and Govindaraj, 2018)

Theorem 3.3. *Suppose A satisfies the condition*

$$|\arg(\lambda(A))| > \frac{\alpha\pi}{2}, \quad (3.9)$$

$f(x, y(x))$ satisfies the condition $f(x, 0) = 0$ $\|f(x, y(x))\| \leq M'\|y\|$ for some $M' > 0$, and $\|g_k(x)\|$ is bounded, Then, the zero solution of the given system(1.1) is asymptotically stable.

Proof. The solution of the system (1.1) with initial condition (1.2) can be written in the general form

$$y(x) = \int_0^x e^{A(x-\tau)} (B\phi(\tau-r)(g_i(\tau)y(\tau) + f(\tau, y(\tau))) d\tau + e^{Ax} c_i, \quad (3.10)$$

Taking norm on both sides, we get the following,

$$\|y(x)\| = \|e^{Ax} c_i\| + \int_0^x \|e^{A(x-\tau)}\| \|B\phi(\tau-r)(g_i(\tau)y(\tau) + f(\tau, y(\tau)))\| d\tau, \quad (3.11)$$

By imposing the given conditions on g_k and $f(x, y(x))$, finally, we have

$$\begin{aligned} \|y(x)\| &\leq \|e^{Ax} c_i\| + \int_0^x \|e^{A(x-\tau)}\| \|B\phi(\tau-r)\| \|g_i(\tau)\| \|y(\tau)\| + \|f(\tau, y(\tau))\| d\tau, \\ &\leq \|e^{Ax} c_i\| + (BCM + M') \int_0^x \|e^{A(x-\tau)}\| \|y(\tau)\| d\tau, \end{aligned}$$

Making use of the Gronwall's inequality, we may reach the simplified form of the above inequality,

$$\leq \|e^{Ax} c_i\| \exp\{N \int_0^x e^{a\tau} d\tau\}$$

where $N = (BKM + M')$ and the norm of ϕ is given by $\|\phi\| = \max_{-1 \leq t \leq 0} \phi(x) = K$. From the Lemma 3.1, $\exp\{N \int_0^x \|e^{a\theta}\| d\theta\}$ is bounded.

Further, from the condition (3.7) we have $\|e^{Ax}\| \rightarrow 0$ as $x \rightarrow \infty$. Finally, we have $\lim_{x \rightarrow \infty} y(x) = 0$. $\square$

4 Example

This section consists of some examples of a non-linear delayed system. Particularly delayed Logistic equation and Delayed Mathieu equations were analyzed in this section. From this it is recognized that the proposed method has some peculiar advantages that is, there is no need of solving transcendental equation even though the nonlinearity with delayed term exists. To show the efficiency these problems were solved in MATLAB by using dde package.

4.1 Analysis of Logistic equations

The evolution of a population with available resources can be established by the eminent model, which is popularly known as the autonomous delayed logistic equation. Consider the logistic population model in the following Cartesian form,

$$y'(x) = ay(x) + by(x-r)y(x), \quad (4.1)$$

$$y(x) = \phi(x), -r \leq x \leq 0, \quad (4.2)$$

where $y(x)$ represents the population density at time t , a represents the fundamental growth rate and b is population carrying ability, Equation (4.1) is also well known as Hutchinson's equation.

Representing the Logistic equation in the standard form, $y'(x) = Ay(x) + By(x - r)y(x) + f(x, y(x))$, where $A = B = -\lambda$, $f(x, y(x)) = 0$,

The eigenvalues is $-\lambda$ which satisfies $|\arg(\text{spec}(A))| > \frac{\pi}{2}$. Since the system (4.1) fulfills all the necessary conditions. Hence according to the Theorem 3.1, the given system is Stable. The stability of delayed logistic equation can be seen in the Fig1.

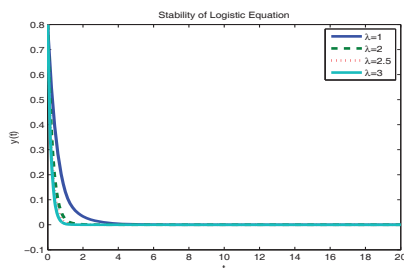


Figure 1: Stability of Logistic Equation.

4.2 Analysis of Mathieu Equations

Mathieu equation has a considerable role in determining the vibration modes of a stretched membrane . The main objective of this work is to analyze the delayed and damped Mathieu equation. The main advantage is that it does not require the analytical solution to analyze the stability.

Consider the Mathieu equation in the following Cartesian form,

$$y''(x) + \epsilon y'(x) + [\delta + \epsilon \alpha \cos(x)]y(x) = by(x - r)y(x), \quad x \in [0, r] \quad (4.3)$$

$$y(x) = \phi(x) \quad x \in [-r, 0] \quad (4.4)$$

δ is the transient curve , ϵ is the amplitude of the parametric resonance, b is the delay amplitude and $r > 0$ is the time delay, and $\phi(x)$ is a given function.

Representing Mathieu equation (4.3) in the standard form, $y'(x) = Ay(x) + By(x - r)y(x) + f(x, y(x))$, where $f(x, y(x)) = (0, \epsilon \alpha \cos(x)y(x))^T$, $B = b$ and

$$A = \begin{bmatrix} 0 & 1 \\ -\delta & -\epsilon \end{bmatrix}.$$

Let us analyze for the values $\epsilon = 3, \delta = 2, \alpha = -1$. For the given values of ϵ, δ, α , the eigenvalues of A are given by -1 and -2 , which satisfies the condition $|\arg(\text{spec}(A))| > \frac{\pi}{2}$. Since the system (4.3) satisfies all the conditions of the Theorem . Hence it is asymptotically stable. See Fig2.

5 Conclusion

New stability criteria is introduced by applying Gronwall inequality and Fixed point theory to analyze the stability of delay differential systems. Sufficient conditions for the stability of the

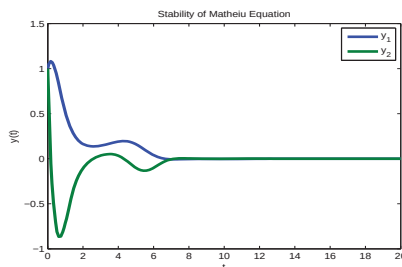


Figure 2: Stability of Mathieu Equation.

delayed equations were derived by employing matrix analysis and nonlinearity conditions. We have considered two different equations, Logistic and Mathieu equations. Stability of the given systems with damping is obtained in this paper. It may be noted that the proposed method gives a notion of applying eigenvalue concepts in analyzing stability of the given system actually without solving the entire system. Especially the solution of delayed equation is very difficult to obtain, hence it has been proven to be a powerful tool even when the nonlinearity exists. Finally, an illustrative examples were presented for the class of delay dynamical system. Future work includes the stability analysis of integro neutral delayed equations.

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