

On evolution of space curve

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ABSTRACT

This paper deals with the evolution of space curve. The change rate of length, curvature and torsion are derived for space curves.

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1 Introduction

In many design and construction problems, such as reverse engineering, multi-component mesh optimization and industrial geometric design from a wireframe, smooth space curves are required to be produced by noisy data. This aim can be significantly done by using geometric flows for blurring (smoothing) the input data with shape preserved (Xu, 2012). In order to restore an original image whose some part of it is corrupted, one can use inpainting. The first step for inpainting is using curve evolution to find the boundary of defected zone. When we are dealing with a 3D defected image that needs inpainting, we can use the curve evolution in space. Therefore, the knowledge on the change rate of the length or other geometric flows are very important. Richardson and King in (Richardson and King, 2002) applied Lie group based similarity methods to the study of class of objects, namely motions of a space curve. In particular, they have scribed the motion of an evolution curve with curvature as velocity flow. Xu in (Xu, 2012) defined some differential operators on space curves. Based on those operators, several second-order and fourth-order geometric flows for evolving space curves have been constructed (For further reading, we can refer to articles (El-Ahmady and Al-Hesiny, 2013) and (Gong, 2013)). In this paper, we generalize the topics such as change rate of length, curvature

from planar curve (that are seen for instance in (Kimmel, 2003)) to the space curve evolution. To this end, consider the family of space curves given by $C(s, t) : [0, L(t)] \times [0, T) \rightarrow \mathbb{R}^3$ where s is the arc length of the curve C at time t and $L(t)$ is the length of the curve at time t . Suppose that the curve evolution equation in time is given by

$$C_t = \vec{V}, \quad C(s, 0) = C_0(s),$$

where $\vec{V}(s, t) : [0, L] \times [0, T) \rightarrow \mathbb{R}^3$ is a velocity vector field that changes smoothly along the curve. The evolution curve can equivalently written in terms of tangential $\vec{T} = C_s$, normal $\vec{N}$ and binormal $\vec{B}$ components of the velocity $\vec{V}$ along the curve by

$$C_t = V^T \vec{T} + V^N \vec{N} + V^B \vec{B}, \quad C(s, 0) = C_0(s), \quad (1.1)$$

where $V^T = \langle \vec{V}, \vec{T} \rangle$, $V^N = \langle \vec{V}, \vec{N} \rangle$ and $V^B = \langle \vec{V}, \vec{B} \rangle$. Of course, the tangential component affects only the parameterization and not the geometric shape of evolving curve. The following lemma confirms the above statement:

Lemma 1.1. *The family of curves $C(p, t)$ that solves the evolution curve (1.1), where V^N and V^B do not depend on the parameterization of the curve, can be converted into the solution of*

$$C_t = V^N \vec{N} + V^B \vec{B}, \quad (1.2)$$

by a change of parameterization.

Proof. We have

$$\frac{\partial}{\partial t} C(p, t) = C_p p_t + C_t.$$

For the length parameterization, we have $C_p = C_s s_p = \vec{T} s_p$. Using these two expressions, we calculate:

$$\frac{\partial}{\partial t} C(p, t) = s_p p_t \vec{T} + V^T \vec{T} + V^N \vec{N} + V^B \vec{B} = (s_p p_t + V^T) \vec{T} + V^N \vec{N} + V^B \vec{B}.$$

By solving the first order ordinary differential equation $s_p p_t + V^T = 0$, we arrive at (1.2).

□

2 Properties of curve evolution

In this section, we describe some properties of the curve evolution. First, the length of a curve is given by $L(t) = \int_0^1 \langle C_p, C_p \rangle^{1/2} dp$. Then,

$$\begin{aligned}
 \frac{d}{dt} L(t) &= \frac{d}{dt} \int_0^1 \langle C_p, C_p \rangle^{1/2} dp = \int_0^1 \frac{\langle C_{tp}, C_p \rangle}{\langle C_p, C_p \rangle^{1/2}} dp \\
 &= \int_0^1 \frac{\langle \frac{\partial}{\partial p} (V^N \vec{N} + V^B \vec{B}), \vec{T}_{s_p} \rangle}{\langle \vec{T}_{s_p}, \vec{T}_{s_p} \rangle^{1/2}} dp \\
 &= \int_0^1 \langle \frac{\partial}{\partial p} (V^N \vec{N} + V^B \vec{B}), \vec{T} \rangle dp \\
 &= \int_0^1 \langle \frac{\partial}{\partial s} (V^N \vec{N} + V^B \vec{B})_{s_p}, \vec{T} \rangle dp \\
 &= \int_0^1 \langle V_s^N \vec{N} + V^N (-\kappa \vec{T} + \tau \vec{B}) + V_s^B \vec{B} + V^B (-\tau \vec{N}), \vec{T} \rangle_{s_p} dp \\
 &= - \int_0^1 s_p \kappa V^N dp = - \int_0^L \kappa V^N ds.
 \end{aligned} \tag{2.1}$$

In above, we used the relation $\frac{\partial}{\partial s} \vec{B} = -\kappa \vec{T} + \tau \vec{B}$.

Now, we calculate the changes of curvature curve in terms of t . Before that, let the metric along the curve be $g = \langle C_p, C_p \rangle^{1/2}$. Then, the change of g in time t is computed as follows:

$$\begin{aligned}
 g_t &= \frac{\langle C_{tp}, C_p \rangle}{\langle C_p, C_p \rangle^{1/2}} = \langle \frac{\partial}{\partial p} (V^N \vec{N} + V^B \vec{B}), \vec{T} \rangle \\
 &= g \langle \frac{\partial}{\partial s} (V^N \vec{N} + V^B \vec{B}), \vec{T} \rangle \\
 &= g \langle V_s^N \vec{N} + V^N (-\kappa \vec{T} + \tau \vec{B}) + V_s^B \vec{B} + V^B (-\tau \vec{N}), \vec{T} \rangle \\
 &= -\kappa g V^N.
 \end{aligned} \tag{2.2}$$

To compute the derivatives of C in p , first it is clear that

$$C_p = g \vec{T}. \tag{2.3}$$

Moreover, we have

$$C_{pp} = g \frac{\partial}{\partial s} C_p = g \frac{\partial}{\partial s} (g \vec{T}) = g g_s \vec{T} + g^2 \kappa \vec{N}. \tag{2.4}$$

Also, the third derivative of C in p is given by

$$\begin{aligned}
 C_{ppp} &= g \frac{\partial}{\partial s} C_{pp} = g \frac{\partial}{\partial s} (g g_s \vec{T} + g^2 \kappa \vec{N}) \\
 &= g g_s^2 \vec{T} + g^2 g_{ss} \vec{T} + 3 g^2 g_s \kappa \vec{N} + g^3 \kappa_s \vec{N} + g^3 \kappa (-\kappa \vec{T} + \tau \vec{B}) \\
 &= (g g_s^2 + g^2 g_{ss} - \kappa^2 g^3) \vec{T} + (3 g^2 g_s \kappa + g^3 \kappa_s) \vec{N} + g^3 \kappa \tau \vec{B}.
 \end{aligned} \tag{2.5}$$

The change of C in terms of t and p is given by

$$\begin{aligned}
 C_{tp} &= \frac{\partial}{\partial p} C_t = \frac{\partial}{\partial p} (V^N \vec{N} + V^B \vec{B}) = g \frac{\partial}{\partial s} (V^N \vec{N} + V^B \vec{B}) \\
 &= g V_s^N \vec{N} + g V^N (-\kappa \vec{T} + \tau \vec{B}) + g V_s^B \vec{B} + g V^B (-\tau \vec{N}) \\
 &= -\kappa g V^N \vec{T} + (g V_s^N - g V^B \tau) \vec{N} + (g V^N \tau + g V_s^B) \vec{B}.
 \end{aligned} \tag{2.6}$$

$$\begin{aligned}
 C_{tpp} &= \frac{\partial}{\partial p} C_{tp} = g \frac{\partial}{\partial s} (-\kappa g V^N \vec{T} + (g V_s^N - g V^B \tau) \vec{N} + (g V^N \tau + g V_s^B) \vec{B}) \\
 &= -g^2 \kappa_s V^N \vec{T} - \kappa g g_s V^N \vec{T} - \kappa g^2 V_s^N \vec{T} - \kappa^2 g^2 V^N \vec{N} + g g_s V_s^N \vec{N} \\
 &\quad + g^2 V_{ss}^N \vec{N} + g^2 V_s^N (-\kappa \vec{T} + \tau \vec{B}) - g g_s V^B \tau \vec{N} - g^2 V_s^B \tau \vec{N} \\
 &\quad - g^2 V^B \tau_s \vec{N} - g^2 V^B \tau (-\kappa \vec{T} + \tau \vec{B}) + g g_s V^N \tau \vec{B} + g^2 V_s^N \tau \vec{B} \\
 &\quad + g^2 V^N \tau_s \vec{B} + g^2 V^N \tau (-\tau \vec{N}) + g g_s V_s^B \vec{B} + g^2 V_{ss}^B \vec{B} + g^2 V_s^B (-\tau \vec{N}) \\
 &= (-g^2 \kappa_s V^N - \kappa g g_s V^N - 2\kappa g^2 V_s^N + g^2 V^B \tau \kappa) \vec{T} \\
 &\quad + (-\kappa^2 g^2 V^N + g g_s V_s^N + g^2 V_{ss}^N - g g_s V^B \tau - g^2 V_s^B \tau - g^2 V^B \tau_s \\
 &\quad - g^2 V^N \tau^2 - g^2 V_s^B \tau) \vec{N} + (g^2 V_s^N \tau - g^2 V^B \tau^2 + g g_s V^N \tau + g^2 V_s^N \tau \\
 &\quad + g^2 V^N \tau_s + g g_s V_s^B + g^2 V_{ss}^B) \vec{B},
 \end{aligned} \tag{2.7}$$

The following relations are also satisfied for the outer products of the derivatives of C in t and p :

$$\begin{aligned}
 C_{tp} \times C_{pp} &= (-\kappa g V^N \vec{T} + (g V_s^N - g V^B \tau) \vec{N} + (g V^N \tau + g V_s^B) \vec{B}) \\
 &\quad \times (g g_s \vec{T} + g^2 \kappa \vec{N}) \\
 &= (-g^3 V^N \tau \kappa - g^3 \kappa V_s^B) \vec{T} + (g^2 V^N \tau g_s + g^2 V_s^B g_s) \vec{N} \\
 &\quad + (-\kappa^2 g^3 V^N - g^2 g_s V_s^N + g^2 V^B \tau g_s) \vec{B},
 \end{aligned} \tag{2.8}$$

$$\begin{aligned}
 C_p \times C_{tpp} &= (-g^3 V_s^N \tau + g^3 V^B \tau^2 - g^2 g_s V^N \tau - g^3 V_s^N \tau - g^3 V^N \tau_s \\
 &\quad - g^2 g_s V_s^B - g^3 V_{ss}^B) \vec{N} + (-\kappa^2 g^3 V^N + g^2 g_s V_s^N + g^3 V_{ss}^N \\
 &\quad - g^3 V_s^B \tau - g^3 V^B \tau_s - g^2 g_s V^B \tau - g^3 V^N \tau^2 - g^3 V_s^B \tau) \vec{B}.
 \end{aligned} \tag{2.9}$$

Now, we are ready to calculate the changes of curvature in time:

$$\begin{aligned}
 \kappa_t &= \frac{\partial}{\partial t} \frac{|C_p \times C_{pp}|}{|C_p|^3} = \frac{\partial}{\partial t} \frac{\langle C_p \times C_{pp}, C_p \times C_{pp} \rangle^{1/2}}{g^3} \\
 &= \frac{\frac{\langle C_{tp} \times C_{pp} + C_p \times C_{tpp}, C_p \times C_{pp} \rangle}{\langle C_p \times C_{pp}, C_p \times C_{pp} \rangle^{1/2}} g^3 - 3g^2 g_t \langle C_p \times C_{pp}, C_p \times C_{pp} \rangle^{1/2}}{g^6}.
 \end{aligned} \tag{2.10}$$

By (2.3) and (2.4), it is seen that

$$C_p \times C_{pp} = g^3 \kappa \vec{B}. \tag{2.11}$$

Moreover, by (2.8), (2.9) and (2.11) we have

$$\begin{aligned}
 \langle C_{tp} \times C_{pp} + C_p \times C_{tpp}, C_p \times C_{pp} \rangle &= -\kappa^3 g^6 V^N - g^5 g_s V_s^N \kappa \\
 &\quad + g^5 g_s V^B \tau \kappa - g^6 V^N \kappa^3 + g^5 g_s V_s^N \kappa + g^6 V_{ss}^N \kappa \\
 &\quad - g^6 V_s^B \tau \kappa - g^6 V^B \tau_s \kappa - g^5 g_s V^B \tau \kappa - g^6 V^N \tau^2 \kappa - g^6 V_s^B \tau \kappa \\
 &= -2\kappa^3 g^6 V^N + g^6 V_{ss}^N \kappa - g^6 V^B \tau_s \kappa - g^6 V^N \tau^2 \kappa - 2g^6 V_s^B \tau \kappa.
 \end{aligned} \tag{2.12}$$

Finally, by (2.10), (2.11) and (2.12) we obtain the following result:

$$\kappa_t = \kappa^2 V^N + V_{ss}^N - V^B \tau_s - V^N \tau^2 - 2V_s^B \tau. \quad (2.13)$$

By (2.13), we see that for planar curves, i.e., $\tau = 0$, we have $\kappa_t = \kappa^2 V^N + V_{ss}^N$.

Now, we want to calculate the change of torsion τ with respect to time. To this end, first the operators $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ can be commuted according to the following rule:

$$\frac{\partial}{\partial t} \frac{\partial}{\partial s} = \kappa V^N \frac{\partial}{\partial s} + \frac{\partial}{\partial s} \frac{\partial}{\partial t}, \quad (2.14)$$

since

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\partial}{\partial s} &= \frac{\partial}{\partial t} \left(\frac{1}{g} \frac{\partial}{\partial p} \right) = -\frac{g_t}{g^2} \frac{\partial}{\partial p} + \frac{1}{g} \frac{\partial}{\partial p} \frac{\partial}{\partial t} \\ &= \frac{\kappa g V^N}{g^2} \frac{\partial}{\partial s} + \frac{1}{g} g \frac{\partial}{\partial s} \frac{\partial}{\partial t} = \kappa V^N \frac{\partial}{\partial s} + \frac{\partial}{\partial s} \frac{\partial}{\partial t}. \end{aligned} \quad (2.15)$$

In view of (2.14),

$$\begin{aligned} \vec{T}_t &= \frac{\partial^2 C}{\partial t \partial s} = \frac{\partial^2 C}{\partial s \partial t} + \kappa V^N \frac{\partial C}{\partial s} = \frac{\partial}{\partial s} (V^N \vec{N} + V^B \vec{B}) + \kappa V^N \vec{T} \\ &= V_s^N \vec{N} + V^N (-\kappa \vec{T} + \tau \vec{B}) + V_s^B \vec{B} + V^B (-\tau \vec{N}) + \kappa V^N \vec{T} \\ &= (V_s^N - \tau V^B) \vec{N} + (V_s^B + \tau V^N) \vec{B}. \end{aligned} \quad (2.16)$$

Furthermore, from

$$\frac{\partial \vec{T}}{\partial s} = \kappa \vec{N},$$

we have

$$\frac{\partial^2 \vec{T}}{\partial t \partial s} = \kappa_t \vec{N} + \kappa \vec{N}_t.$$

Therefore, one can write:

$$\begin{aligned} \kappa V^N \frac{\partial \vec{T}}{\partial s} + \frac{\partial}{\partial s} ((V_s^N - \tau V^B) \vec{N} + (V_s^B + \tau V^N) \vec{B}) \\ = (\kappa^2 V^N + V_{ss}^N - V^B \tau_s - V^N \tau^2 - 2V_s^B \tau) \vec{N} + \kappa \vec{N}_t \end{aligned} \quad (2.17)$$

In view of (2.17), we have:

$$\begin{aligned} \kappa^2 V^N \vec{N} + V_{ss}^N \vec{N} - V^B \tau_s \vec{N} - \tau V_s^B \vec{N} - V_s^N \kappa \vec{T} \\ + V^B \tau \kappa \vec{T} + \tau V_s^N \vec{B} - \tau^2 V^B \vec{B} + V_{ss}^B \vec{B} + \tau_s V^N \vec{B} \\ + \tau V_s^B \vec{B} - V_s^B \tau \vec{N} - \tau^2 V^N \vec{N} \\ = \kappa^2 V^N \vec{N} + V_{ss}^N \vec{N} - V^B \tau_s \vec{N} - \tau V^N \tau^2 \vec{N} - 2V_s^B \tau \vec{N} + \kappa \vec{N}_t, \end{aligned} \quad (2.18)$$

that implies

$$\vec{N}_t = (\tau V^B - V_s^N) \vec{T} + \frac{\tau V_s^N - \tau^2 V^B + V_{ss}^B + \tau_s V^N + \tau V_s^N}{\kappa} \vec{B}, \quad (2.19)$$

and

$$\vec{B}_t = (-\tau V^N - V_s^B) \vec{T} + \frac{-\tau V_s^N + \tau^2 V^B - V_{ss}^B - \tau_s V^N - \tau V_s^N}{\kappa} \vec{N}. \quad (2.20)$$

Now, from $\frac{\partial \vec{B}}{\partial s} = -\tau \vec{N}$, we have

$$\frac{\partial^2 \vec{B}}{\partial t \partial s} = -\tau_t \vec{N} - \tau \vec{N}_t. \quad (2.21)$$

The equality (2.22) yields that

$$\begin{aligned} \kappa V^N \frac{\partial \vec{B}}{\partial s} + \frac{\partial}{\partial s}((-V_s^B - \tau V^N) \vec{T} + \frac{-\tau V_s^N + \tau^2 V^B - V_{ss}^B - \tau_s V^N - \tau V_s^N}{\kappa} \vec{N}) \\ = -\tau_t \vec{N} - \tau((\tau V^B - V_s^N) \vec{T} + \frac{\tau V_s^N - \tau^2 V^B + V_{ss}^B + \tau_s V^N + \tau V_s^N}{\kappa} \vec{B}). \end{aligned} \quad (2.22)$$

Thus, the change of torsion in time t is as follows:

$$\tau_t = 2\kappa\tau V^N + \kappa V_s^B - \frac{\partial}{\partial s} \left(\frac{-\tau V_s^N + \tau^2 V^B - V_{ss}^B - \tau_s V^N - \tau V_s^N}{\kappa} \right). \quad (2.23)$$

In the case that $V^N = \kappa$ and $V^B = 0$, i.e., we are dealing with curvature flow, the following theorem can be stated on the $A(t)$ where A is the area of the minimal disk bounded by a closed curve in space evolving by

$$\frac{\partial C}{\partial t} = \kappa \vec{N}.$$

Theorem 2.1. (?) *Let $A(t)$ be the area of the minimal disk bounded by a closed curve in space evolving by the curvature flow, then $A'(t) \leq -2\pi$.*

3 Discussion on flows

In this section, we discuss on different flows. First, for the case $V^N = 1$, $V^B = 0$, by (2.1) we have (Kimmel, 2003)

$$\frac{dL}{dt} = - \int_0^L \kappa ds = -2\pi,$$

which means that the change rate of length is constant. Moreover, by (2.13) and (2.23) the change rate of curvature and torsion in time are given by

$$\kappa_t = \kappa^2 - \tau^2, \quad (3.1)$$

and

$$\tau_t = 2\kappa\tau - \frac{\partial}{\partial s} \left(\frac{-\tau_s}{\kappa} \right) = 2\kappa\tau + \frac{\tau_{ss}\kappa - \kappa_s\tau_s}{\kappa^2}. \quad (3.2)$$

By (3.1), it is seen that if $\tau = 0$, then we will have a Riccati equation whose solution is given by

$$\kappa(p, t) = \frac{\kappa(p, 0)}{1 - t\kappa(p, 0)},$$

with singularity at $t = 1/\kappa(p, 0)$. The equation (3.2) shows that the torsion can blow up even though the curvature remains bounded. For $V^N = 0, V^B = 1$, the constant evolution equation reads

$$C_t = \vec{B}.$$

In this case, by (2.1), during the evolution, the length of the curve is constant. Moreover, the curvature change in time is given by $\kappa_t = -\tau_s$. Moreover, the torsion change, in time, is given by

$$\tau_t = -\frac{\partial}{\partial s}\left(\frac{\tau^2}{\kappa}\right).$$

4 Numerical scheme

In this section, we present an implicit finite difference scheme for solving the system given by (3.1) and (3.2).

Example 1. In the following space curve, we show how the rate of curvature and torsion is changed.

$$C(s, t) = \{e^t(2 + \sin(3s))\cos(s), e^t(2 + \sin(3s))\sin(s), t\cos(3s)\}.$$

By forward difference scheme in time and spatial domains we have:

$$\kappa_{i,j+1} = \kappa_{i,j} + \Delta t(\kappa_{i,j}^2 - \tau_{i,j}^2), \quad (4.1)$$

and

$$\begin{aligned} & -\frac{\kappa_{i,j+1}}{\Delta^2 x} \tau_{i-1,j+1} + \left(\frac{\kappa_{i,j+1}^2}{\Delta t} - 2\kappa_{i,j+1}^3 + 2\frac{\kappa_{i,j+1}}{\Delta^2 x} - \frac{\kappa_{i+1,j+1} - \kappa_{i,j+1}}{\Delta^2 x}\right) \tau_{i,j+1} \\ & + \left(-\frac{\kappa_{i,j+1}}{\Delta^2 x} + \frac{\kappa_{i+1,j+1} - \kappa_{i,j+1}}{\Delta^2 x}\right) \tau_{i+1,j+1} = \frac{\kappa_{i,j+1}^2}{\Delta t} \tau_{i,j}. \end{aligned} \quad (4.2)$$

By computing $\kappa_{i,j+1}$ from relation (4.1) and replacing it in relation (4.2), we obtain an upper triangular system from which the torsion $\tau_{i,j+1}$ is determined. Figure (1) shows the change of curvature and torsion at times 0.1, 0.4, 0.8, 1, respectively.

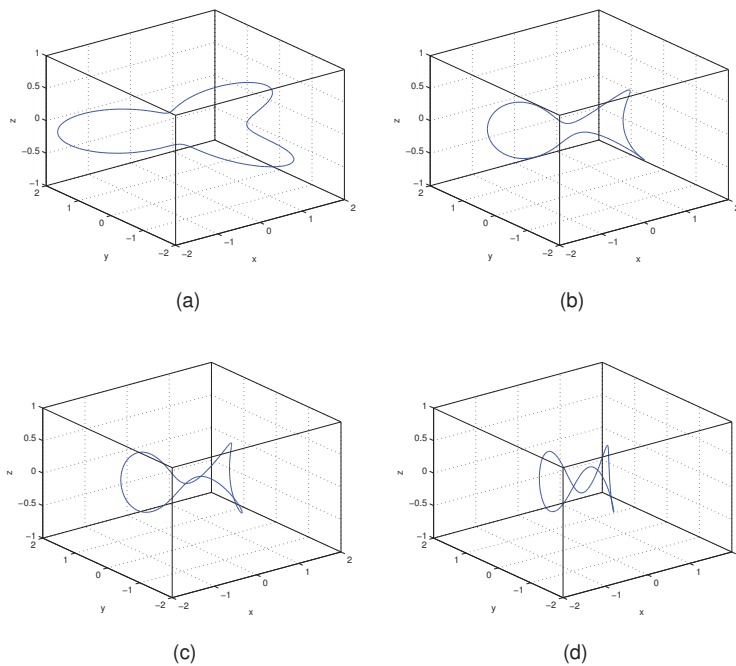


Figure 1: The change of curvature and torsion in different times.

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