

The Balakrishnan Alpha Skew Logistic Distribution: Properties and Applications

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ABSTRACT

In this paper an attempt is made to develop a new bimodal alpha skew logistic distribution under Balakrishnan (2002) mechanism. Some of its distributional as well as moments properties are studied. Some of the extensions of this new distribution are also briefly introduced. Finally, suitability of the proposed distribution is checked with the help of data fitting examples of real life.

Keywords: Skew Normal distribution, Alpha Skew Logistic Distribution, Bimodal Distribution, Balakrishnan Skew Normal distribution, AIC, BIC, Likelihood Ratio Test.

Maths Classification: Primary: 62E15, 60E05; Secondary: 62P05

1. INTRODUCTION

Although one cannot undermine the applications and the value of normal distribution in statistics but there are several random phenomena in nature that we sometimes encounter in datasets having high values of skewness and kurtosis, bimodal and multimodal behaviours in their frequency curve which can't be described by unimodal normal distribution (for details see Hammel et al. (1983), Dimitrov et al. (1997), Sinha (2012), Chakraborty et al. (2015), among others).

To deal with such type of random phenomena, Azzalini (1985) first introduced the path breaking skew normal distribution in his seminal paper entitled "A class of distributions which includes the normal ones. *Scandinavian Journal of Statistics*, 171-178" and its probability density function (pdf) is given by

$$f_Z(z; \lambda) = 2\varphi(z)\Phi(\lambda z); \quad -\infty < z < \infty \quad (1)$$

where, φ and Φ are, respectively, the pdf and cumulative distribution function (cdf) of standard normal variable Z and $\lambda \in R$, the skewness parameter. Symbolically, denoted by $SN(\lambda)$.

After the introduction of this distribution, a lot of research studies are carried out to introduce different skew normal distributions (for details see Chakraborty and Hazarika, 2011). In 2002, Balakrishnan introduced the generalization of $SN(\lambda)$ distribution with the pdf given by

$$f_Z(z; \lambda, n) = \varphi(z)[\Phi(\lambda z)]^n / C_n(\lambda); \quad -\infty < z < \infty, \lambda \in R \quad (2)$$

where, n , a positive integer and $C_n(\lambda) = E[\Phi^n(\lambda U)]$, $U \sim N(0,1)$. For $\lambda=1$, it reduces to $SN(\lambda)$ in equation (1). This distribution is known as Balakrishnan skew normal distribution. There are different extensions and generalizations of this distribution (for details see Sharafi and Behboodian (2008), Yadegari et al. (2008), Bahrami et al. (2009), Hasanaliipour and Sharafi (2012) and among others).

Instead of using the cdf $\Phi(\cdot)$ in equation (1), Huang and Chen (2007) introduced a new concept of skew function $G(\cdot)$, a Lebesgue measurable function with $0 \leq G(z) \leq 1$ and $G(z) + G(-z) = 1$, $z \in R$, almost everywhere. The corresponding pdf is given by

$$f(z) = 2h(z)G(z); z \in R \quad (3)$$

Employing this idea Elal-Olivero (2010) developed alpha skew normal distribution having pdf

$$f(z; \alpha) = \left(\frac{(1 - \alpha z)^2 + 1}{2 + \alpha^2} \right) \varphi(z); z \in R, \alpha \in R \quad (4)$$

The different generalization of the pdf in equation (4) were studied and analyzed (for details see Venegas et al. (2016), Louzada et al. (2017), Sharafi et al. (2017) etc.). A new form of skew logistic distribution and its generalization was given by Chakraborty et al. (2012) and Hazarika and Chakraborty (2015), respectively. Using exactly the similar approach of Elal-Olivero (2010), Harandi and Alamatsaz (2013) and Hazarika and Chakraborty (2014) investigated a class of alpha skew Laplace distributions and alpha skew logistic distributions, respectively.

The aim of this article is to propose a new alpha skew logistic distribution following the idea of Balakrishnan (2002) mechanism and some of its distributional properties are investigated. It is shown that this distribution (known as Balakrishnan alpha skew logistic distribution denoted by $BASLG_2(\alpha)$ where, $\alpha \in R$) is flexible enough to support both uni-modal and bi-modal behaviours as well as positive and negative skewness. For some real datasets, the $BASLG_2(\alpha)$ distribution is found to give better fitting when compared to some other known distributions.

The contents of the rest of this article are organized as follows. In section 2, we introduce a new alpha skew Logistic $BASLG_2(\alpha)$ distribution and discuss some of its important distributional properties. Section 3 discusses about some of the extensions of $BASLG_2(\alpha)$ distribution. In section 4, the maximum likelihood estimators of the parameters of location and scale extension of $BASLG_2(\alpha)$ distribution, which is now denoted by $BASLG_2(\alpha, \mu, \beta)$, are derived and some numerical examples based on real life datasets are provided to show the applicability of the proposed model. Finally, conclusions are provided in section 5.

2. A NEW ALPHA SKEW LOGISTIC DISTRIBUTION

In this section we define a new alpha skew logistic distribution and discuss some of its distributional properties as follows

Definition 1: A continuous random variable Z with the pdf given as

$$f_Z(z; \alpha) = \frac{[(1 - \alpha z)^2 + 1]^2}{C_2(\alpha)} \frac{e^{-z}}{(1 + e^{-z})^2}; z \in R \quad (5)$$

is said to follow Balakrishnan alpha skew logistic distribution with skewness parameter $\alpha \in R$. In the rest of this article we shall refer the distribution in equation (5) as $BASLG_2(\alpha)$. The normalizing constant $C_2(\alpha) > 0$ is computed as follows:

$$\begin{aligned} C_2(\alpha) &= \int_{-\infty}^{\infty} [(1 - \alpha z)^2 + 1]^2 \frac{e^{-z}}{(1 + e^{-z})^2} dz \\ &= \int_{-\infty}^{\infty} (4 - 8\alpha z + 8\alpha^2 z^2 - 4\alpha^3 z^3 + \alpha^4 z^4) \frac{e^{-z}}{(1 + e^{-z})^2} dz \end{aligned}$$

$$\begin{aligned}
 &= 4 \int_{-\infty}^{\infty} \frac{e^{-z}}{(1+e^{-z})^2} dz - 8\alpha \int_{-\infty}^{\infty} \frac{ze^{-z}}{(1+e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^{\infty} \frac{z^2 e^{-z}}{(1+e^{-z})^2} dz - 4\alpha^3 \int_{-\infty}^{\infty} \frac{z^3 e^{-z}}{(1+e^{-z})^2} dz + \alpha^4 \int_{-\infty}^{\infty} \frac{z^4 e^{-z}}{(1+e^{-z})^2} dz \\
 &= 4 + \frac{1}{3} \pi^2 \alpha^2 \left(8 + \frac{7}{3} \pi^4 \alpha^4 \right)
 \end{aligned}$$

2.1. Some particular cases:

- When $\alpha = 0$, we get the standard logistic distribution given by

$$f(z) = \frac{e^{-z}}{(1+e^{-z})^2} \quad (6)$$

and is denoted by $Z \rightarrow LG(0, 1)$.

- When $\alpha \rightarrow \pm\infty$, we get a Bimodal Logistic (BLG) distribution (see Hazarika and Chakraborty, 2014) given by

$$f(z) = \frac{15 z^4}{7 \pi^4} \frac{e^{-z}}{(1+e^{-z})^2} \quad (7)$$

and is denoted by $Z \xrightarrow{d} BLG(4)$.

- If $Z \sim BASLG_2(\alpha)$ then $-Z \sim BASLG_2(-\alpha)$

2.2. Plots of pdf:

The plot of the pdf of $BASLG_2(\alpha)$ distribution is shown in figure 1 for studying the variation in its shape with respect to the parameter α which clearly shows the role of α plays in skewness and kurtosis of the pdf. It's observed that the skewness is positive (negative) according as $\alpha > (<) 0$ while for $\alpha = 0$ its symmetrical.

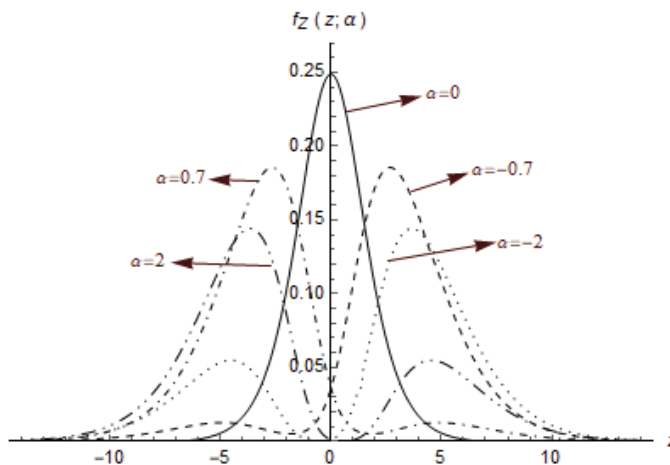


Figure 1: Plots of pdf of $BASLG_2(\alpha)$

Theorem 1: Let $Z \sim BASLG_2(\alpha)$ then its cdf is given by

$$F_Z(z; \alpha) = \frac{1}{(1+e^z)C_2(\alpha)} \left((2+z\alpha(-2+z\alpha))(e^z(2+z\alpha(-2+z\alpha)) - 4(1+e^z)\alpha(-1+z\alpha)\text{Log}[1+e^z]) - 4(1+e^z)\alpha^2((4+3z\alpha(-2+z\alpha))Li_2[-e^z] + 6\alpha((1-z\alpha)Li_3[-e^z] + \alpha Li_4[-e^z])) \right) \quad (8)$$

Where $Li_n[z] = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ is the poly-logarithm function (Prudnikov et al., 1986).

Proof: see Appendix A.

In particular, when $\alpha \rightarrow \pm\infty$, $F_Z(z; \alpha)$ in equation (8) tends to the cdf of BLG(4) distribution as

$$F_Z(z) = \frac{15}{7\pi^4} \left(\frac{e^z z^4}{1+e^z} - 4z^3 \text{Log}[1+e^z] - 12z^2 \text{Li}_2[-e^z] + 24z \text{Li}_3[-e^z] - 24 \text{Li}_4[-e^z] \right) \quad (9)$$

2.3. Plots of cdf:

The plots of the cdf of $BASLG_2(\alpha)$ distribution in figure 2 exhibit the different variation in its shape with respect to the parameter α .

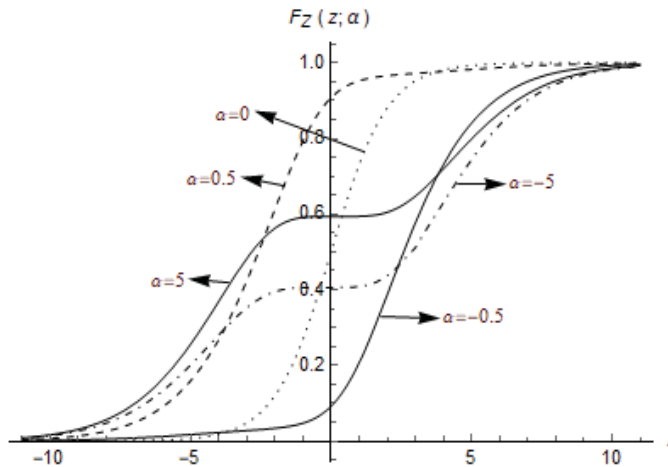


Figure 2: Plots of cdf of $BASLG_2(\alpha)$

From the above figure it can be easily seen that for the values of the parameter $\alpha \neq 0$, the tails get stretched, more over for larger value of $|\alpha|$ the bimodality is prominently expressed (see also figure 1).

2.4. Shapes:

The shapes of $f_Z(z; \alpha)$ can be derived analytically through the study of its first- and second-order derivatives.

$$\begin{aligned} \text{Log}[f_Z(z; \alpha)] &= \text{Log}[(1 - \alpha z)^2 + 1] - \text{Log}[C_2(\alpha)] + \text{Log}[e^{-z}] - \text{Log}[(1 + e^{-z})^2] \\ &= 2 \text{Log}[(1 - \alpha z)^2 + 1] - \text{Log}[C_2(\alpha)] - z - 2 \text{Log}[1 + e^{-z}] \end{aligned} \quad (10)$$

Taking the first-order derivative of $\text{Log}[f_Z(z; \alpha)]$, and equating to zero:

$$\Rightarrow \frac{f_Z'(z; \alpha)}{f_Z(z; \alpha)} = -\frac{4\alpha(1 - \alpha z)}{[(1 - \alpha z)^2 + 1]} - 1 + \frac{2 \exp(-z)}{1 + \exp(-z)}$$

Suppose $z = z_0$ is one of the root of (10) then it corresponds to local maximum, a local minimum or a point of inflexion depending on whether $\omega(z_0) < 0$, $\omega(z_0) > 0$ or $\omega(z_0) = 0$, where $\omega(z_0) = d^2 \text{Log}[f_Z(z; \alpha)] / dz^2$, that is,

$$= \left[-\frac{4\alpha(1 - \alpha z)}{[(1 - \alpha z)^2 + 1]} + \frac{2 \exp(-z)}{1 + \exp(-z)} - 1 \right] f_Z'(z; \alpha) + \left[-\frac{4\alpha^3 z(-2 + \alpha z)}{[(1 - \alpha z)^2 + 1]^2} - \frac{2 \exp(z)}{(1 + \exp(z))^2} \right] f_Z(z; \alpha)$$

In theorem 2, we have shown that the proposed model has at most two modes using a numerical approach.

Theorem 2: The $BASLG_2(\alpha)$ distribution has at most two modes.

Proof: To solve the eqn. (10) is not analytically tractable but plotting the contour plot of eqn. (10), one can easily shown that it has at most three zeros (see figure 3). Again, if the eqn. (10) has three zeros it implies that there are at most two modes (maxima) and at most one anti-mode (minima).

The first differentiation of the pdf of $BASLG_2(\alpha)$ distribution with respect to z gives

$$Df_Z(z; \alpha) = -\frac{e^z \left((1 - \alpha z)^2 + 1 \right) \left(-2 + 2\alpha(z + 2) - z\alpha^2(z + 4) + e^z (2 - 2\alpha(z - 2) + \alpha^2 z(z - 4)) \right)}{(1 + e^z)^3 C_2(\alpha)} \quad (11)$$

Then the contour of the equation $Df_Z(z; \alpha) = 0$ is drawn in figure 3. From the figure, it can be seen that there is at most three zeros of $Df_Z(z; \alpha)$ which implies that $BASLG_2(\alpha)$ distribution has at most two modes. Further from the figure 3, it can also be noted that approximately for $-0.48 < \alpha < 0.48$, $BASLG_2(\alpha)$ distribution remains unimodal.

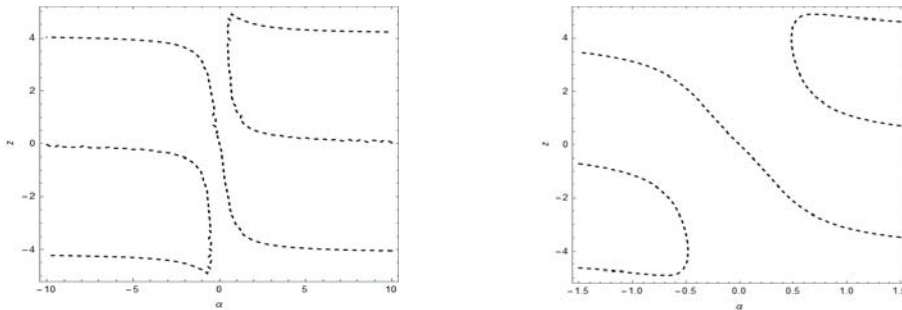


Figure 3: Contour plots of the equation $Df_Z(z; \alpha) = 0$

Theorem 3: Let $Z \sim BASLG_2(\alpha)$ then its moment generating function (mgf) is given by

$$M_Z(t) = \frac{1}{C_2(\alpha)} \left(\pi Csc(\pi) (-8\alpha + 12\pi^2 \alpha^3 + t(4 - 8\pi^2 \alpha^2 + \pi^4 \alpha^4)) + 4\pi\alpha((-4\alpha + \pi^2 \alpha^3 + t(2 - \pi^2 \alpha^2))Cot(\pi) + \pi\alpha(-6\alpha + t(4 - 5\pi^2 \alpha^2)) + 6\pi(t - \alpha)\alpha Cot(\pi))Csc(\pi)^2 + 6\pi^3 t \alpha^3 Csc(\pi)^4 \right) \quad (12)$$

provided $-1 < t < 1$.

Proof: see Appendix B.

In particular, if $\alpha \rightarrow \pm\infty$, $M_Z(t)$ in equation (12) tends to the mgf of $BLG(4)$ distribution as

$$M_Z(t) = \frac{15}{56} Csc(\pi)^5 (115\pi + \pi(76Cos(2\pi) + Cos(4\pi)) - 88Sin(2\pi) - 4Sin(4\pi)) \quad (13)$$

Theorem 4: Let $Z \sim BASLG_2(\alpha)$ then for $k \in N$, the k^{th} moment is given by

$$E(Z^{2k}) = \frac{1}{C_2(\alpha)} 2 \left[4 \left(1 - \frac{1}{2^{k-1}} \right) \Gamma(k+1) \zeta(k) + 8\alpha^2 \left(1 - \frac{1}{2^{k+1}} \right) \Gamma(k+3) \zeta(k+2) + \alpha^4 \left(1 - \frac{1}{2^{k+3}} \right) \Gamma(k+5) \zeta(k+4) \right] \quad (14)$$

and,

$$E(Z^{2k-1}) = \frac{1}{C_2(\alpha)} \left[2 \left(-8\alpha \left(1 - \frac{1}{2^k} \right) \Gamma(k+2) \zeta(k+1) - 4\alpha^3 \left(1 - \frac{1}{2^{k+2}} \right) \Gamma(k+5) \zeta(k+3) \right) \right] \quad (15)$$

where, $\zeta(s) = \sum_{j=1}^{\infty} j^{-s}$ is the Riemann zeta functions and $\Gamma(s) = \int_0^{\infty} t^{s-1} \exp(-t) dt$ (Prudnikov et al., 1986).

Proof: See Appendix C.

In particular it's easy to derive the following expressions for first four moments and variance:

$$E(Z) = \frac{4\pi^2\alpha(10 + 7\pi^2\alpha^2)}{60 + 40\pi^2\alpha^2 + 7\pi^4\alpha^4}$$

$$E(Z^2) = \frac{\pi^2(140 + 392\pi^2\alpha^2 + 155\pi^4\alpha^4)}{7(60 + 40\pi^2\alpha^2 + 7\pi^4\alpha^4)}$$

$$E(Z^3) = \frac{4\pi^4\alpha(98 + 155\pi^2\alpha^2)}{7(60 + 40\pi^2\alpha^2 + 7\pi^4\alpha^4)}$$

$$E(Z^4) = \frac{\pi^4(196 + 1240\pi^2\alpha^2 + 889\pi^4\alpha^4)}{7(60 + 40\pi^2\alpha^2 + 7\pi^4\alpha^4)}$$

$$Var(Z) = \frac{\pi^2(8400 + 17920\pi^2\alpha^2 + 10280\pi^4\alpha^4 + 3456\pi^6\alpha^6 + 1085\pi^8\alpha^8)}{7(60 + 40\pi^2\alpha^2 + 7\pi^4\alpha^4)}$$

The ratio between variance and mean is given in figure (4). From the figure, it is shown that for $\alpha < 0$, the mean is positive and for $\alpha > 0$, the mean is negative though in both the cases the variance is greater than the mean.

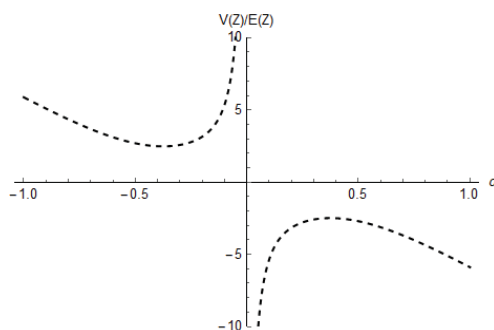


Figure 4: Plots of ratio of variance and mean.

The skewness and kurtosis of $BASLG_2(\alpha)$ distribution can be easily obtained (given in the Appendix E).

Remark 1: By numerically optimizing mean, variance, skewness and the kurtosis with respect to α , we obtain their bounds respectively as $-2.9077 \leq E(Z) \leq 2.9077$, $3.28987 \leq Var(Z) \leq 31.2202$, $0 \leq \beta_1 \leq 1.3945$ and $1.81315 \leq \beta_2 \leq 6.87571$.

The plots of mean, variance, skewness and kurtosis of $BASLG_2(\alpha)$ distribution are shown, respectively, in figure 5, 6, 7 and 8 further validate the Remark 1.

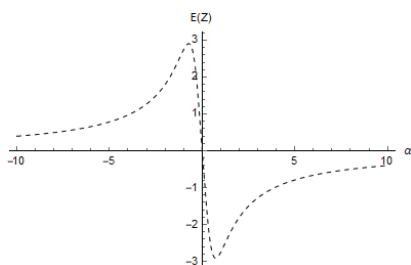


Figure 5: Plot of mean

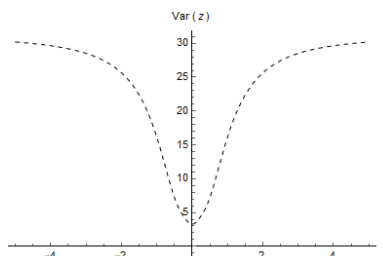


Figure 6: Plot of variance

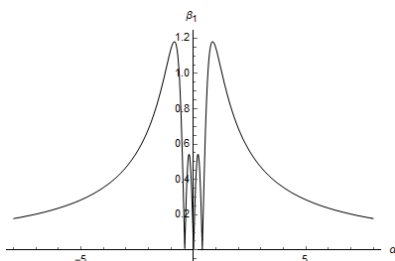


Figure 7: Plot of skewness

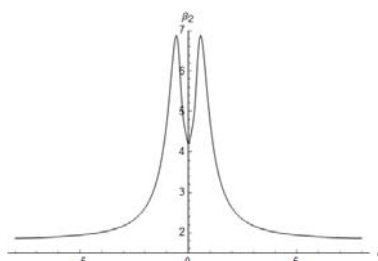


Figure 8: Plot of kurtosis

Remark 2: The pdf of equation (5) can be expressed as sum of two functions, as shown below

$$\begin{aligned} f_Z(z; \alpha) &= \frac{[(1 - \alpha z)^2 + 1]^2}{C_2(\alpha)} \frac{e^{-z}}{(1 + e^{-z})^2} \\ &= \frac{4 + 8\alpha^2 z^2 + \alpha^4 z^4}{C_2(\alpha)} \frac{e^{-z}}{(1 + e^{-z})^2} - \frac{8\alpha z + 4\alpha^3 z^3}{C_2(\alpha)} \frac{e^{-z}}{(1 + e^{-z})^2} \end{aligned}$$

(16)

In equation (16), the first part is symmetric and the second part is asymmetric one. The symmetric part, which is defined below, is symbolically denoted by $SCBASLG_2(\alpha)$.

Definition 2: A continuous random variable Z with the following pdf

$$f_1(z; \alpha) = \frac{4 + 8\alpha^2 z^2 + \alpha^4 z^4}{C_2(\alpha)} \frac{e^{-z}}{(1 + e^{-z})^2}; \quad z \in R, \alpha \in R \quad (17)$$

is the symmetric-component random variable of the model $BASLG_2(\alpha)$. We write it as $Z \sim SCBASLG_2(\alpha)$.

Properties of $SCBASLG_2(\alpha)$:

- (i) If $\alpha = 0$, then $Z \sim LG(0, 1)$.
- (ii) If $\alpha \rightarrow \pm\infty$, then $Z \xrightarrow{d} BLG$

Theorem 5: Let $Z \sim SCBASLG_2(\alpha)$ then its cdf is given by

$$F_1(z; \alpha) = \frac{1}{C_2(\alpha)(1 + e^z)} \left(e^z (4 + 8\alpha^2 z^2 + \alpha^4 z^4) - 4(1 + e^z) \alpha^2 z (4 + \alpha^2 z^2) \text{Log}[1 + e^z] - \right. \\ \left. 4(1 + e^z) \alpha^2 ((4 + 3\alpha^2 z^2) Li_2[-e^z] + 6\alpha^2 (-z Li_3[-e^z] + Li_4[-e^z])) \right) \quad (18)$$

Proof: see Appendix F.

Theorem 6: Let $Z \sim SCBASLG_2(\alpha)$ then its mgf is given by

$$M_Z(t; \alpha) = \frac{\pi \text{Csc}[\pi]}{8C_2(\alpha)(1 + e^z)} \left(\frac{12t + 40\pi^2 \alpha^2 t + 115\pi^4 \alpha^4 t + 4t \text{Cos}[2\pi](-4 - 8\pi^2 \alpha^2 + 19\pi^4 \alpha^4) +}{t \text{Cos}[4\pi](4 - 8\pi^2 \alpha^2 + \pi^4 \alpha^4) - 32\pi \alpha^2 \text{Sin}[2\pi](1 + 2.75\pi^2 \alpha^2) +} \right. \\ \left. 4\pi \alpha^2 \text{Sin}[4\pi](4 - \pi^2 \alpha^2) \right) \quad (19)$$

provided, $-1 < t < 1$

Proof: see Appendix G.

Remark 3: The algorithm here can be used to simulate random numbers for $BASLG_2(\alpha)$ distribution:

Let $g(x; \alpha)$, the density function of $X \sim BASLG_2(\alpha)$ and $g_1(x; \alpha)$, the density function of $Y \sim SCBASLG_2(\alpha)$, with

$$S = \sup_x \frac{g(x; \alpha)}{g_1(x; \alpha)} = \frac{3 + 2\sqrt{2}}{3}$$

Then a random variable from $BASLG_2(\alpha)$ distribution can be generated in the following steps:

Let X and Y are two independent random variables then

- (i) Generate $Y \sim SCBASLG_2(\alpha)$ and $U \sim \text{Uniform}(0,1)$, and they are independent.
- (ii) If $U < \frac{1}{S} \frac{g(x; \alpha)}{g_1(x; \alpha)}$, and set $X=Y$; otherwise go back to step(i) and continue the process.

3. SOME EXTENSIONS OF $BASLG_2(\alpha)$ DISTRIBUTION

In this section we briefly mention some of the possible extensions of $BASLG_2(\alpha)$ distributions. These are being currently investigated and will be reported later.

3.1. The Bivariate $BASLG_2(\alpha)$ Distribution

In this section, using the idea of Louzada et al. (2017) a natural bivariate form of $BASLG_2(\alpha)$ distribution is defined as follows:

Definition 3: A two-dimensional random variable $\mathbf{Z} = (Z_1, Z_2)$ with the following joint pdf

$$f_{\mathbf{Z}}(\mathbf{z}; \alpha, \alpha_1, \alpha_2) = \frac{[(1 - \alpha_1 z_1 - \alpha_2 z_2)^2 + 1]^2}{C_2(\alpha, \alpha_1, \alpha_2)} \Psi(z_1, z_2, \alpha) \quad (20)$$

where, $C_2(\alpha, \alpha_1, \alpha_2)$ and $\Psi(z_1, z_2, \alpha)$ are given in Appendix D; $z_1, z_2, \alpha, \alpha_1, \alpha_2 \in R$. $\Psi(z_1, z_2, \alpha)$ is the pdf of bivariate logistic distribution (see Gumbel, 1961) is said to be Bivariate Balakrishnan alpha skew logistic distribution with the parameters α, α_1 and α_2 , and is denoted by $BBASLG_2(\alpha, \alpha_1, \alpha_2)$.

3.2. A Two- parameter $BASLG_2(\alpha)$ Distribution

Like Bahrami et al. (2009), here we present the definition and some simple properties of a two-parameter Balakrishnan alpha skew normal distribution.

Definition 4: If a continuous random variable Z has the following pdf

$$f_Z(z; \alpha_1, \alpha_2) = \frac{[(1 - \alpha_1 z)^2 + 1]^2 [(1 - \alpha_2 z)^2 + 1]^2}{C_2(\alpha_1, \alpha_2)} \frac{e^{-z}}{(1 + e^{-z})^2}; z \in R \quad (21)$$

where, $C_2(\alpha_1, \alpha_2)$ is given in the Appendix D, it is said to follow two-parameter Balakrishnan alpha skew logistic distribution with parameters $\alpha_1, \alpha_2 \in R$, and is denoted by $TPBASLG_2(\alpha_1, \alpha_2)$.

3.3. Balakrishnan Alpha-Beta Skew Logistic Distribution

In this section, following the study of Shafiei et al. (2016), we present the definition and some simple properties of Balakrishnan alpha-beta skew logistic distribution.

Definition 5: A continuous random variable Z with the following pdf

$$f_Z(z; \alpha) = \frac{[(1 - \alpha z - \beta z^3)^2 + 1]^2}{C_2(\alpha, \beta)} \frac{e^{-z}}{(1 + e^{-z})^2}; z \in R \quad (22)$$

where, $C_2(\alpha, \beta)$ is given in the Appendix D then, it is said to follow Balakrishnan alpha-beta skew logistic distribution with parameters $\alpha, \beta \in R$, and is denoted by $BABSLG_2(\alpha, \beta)$.

3.4. The Log-Balakrishnan Alpha Skew Logistic Distribution

Here, we present the definition and some simple properties of log-Balakrishnan alpha skew logistic distribution similar to the study of Venegas et al. (2016).

Let $Z = e^Y$, then $Y = \text{Log}(Z)$, therefore, the density function of Z is defined as follows:

Definition 6: A continuous random variable Z with the following pdf

$$f_Z(z; \alpha) = \frac{[(1 - \alpha \text{Log}[z])^2 + 1]^2}{C_2(\alpha)} \frac{e^{-\text{Log}[z]}}{z(1 + e^{-\text{Log}[z]})^2}; z > 0 \quad (23)$$

where $C_2(\alpha)$ is given in the Appendix D then, it is said to follow log-Balakrishnan alpha skew logistic distribution with parameter $\alpha \in \mathbb{R}$, and is denoted by $LBASLG_2(\alpha, \beta)$.

4. DATA MODELLING APPLICATIONS

4.1. Location and Scale Extension

If $Z \sim BASLG_2(\alpha)$ then for $\mu \in \mathbb{R}$ and $\beta > 0$, set $Y = \beta Z + \mu$. Therefore, the location - scale extension of $BASLG_2(\alpha)$ distribution has the following pdf given by

$$f_Y(y; \mu, \beta, \alpha) = \frac{1}{\beta C_2(\alpha)} \frac{1}{\left(1 + e^{-\frac{y-\mu}{\beta}}\right)^2} \left[\left(1 - \alpha \left(\frac{y-\mu}{\beta}\right)\right)^2 + 1 \right]^2 e^{-\left(\frac{y-\mu}{\beta}\right)}; y, \mu, \alpha \in \mathbb{R}, \beta > 0 \quad (24)$$

Symbolically, we write, $Y \sim BASLG_2(\alpha, \mu, \beta)$.

4.2. Maximum Likelihood Estimation

Let $Y \sim BASLG_2(\alpha, \mu, \beta)$ then for a random sample $y_1, y_2, \dots, y_n$ of size n , the log likelihood function is given by

$$l(\theta) = 2 \sum_{i=1}^n \log \left[\left\{ 1 - \alpha \left(\frac{y_i - \mu}{\beta} \right) \right\}^2 + 1 \right] - n \log(60 + 40\pi^2 \alpha^2 + 7\pi^4 \alpha^4) - n \log[15] - n \log(\beta) - \sum_{i=1}^n \frac{(y_i - \mu)}{\beta} - 2 \sum_{i=1}^n \text{Log} \left[1 + \text{Exp} \left[\frac{-(y_i - \mu)}{\beta} \right] \right] \quad (25)$$

Where, $\theta = (\alpha, \mu, \beta)$

Differentiating this eqn. (25) above partially with respect to the parameters α , μ and β , the following likelihood equations are obtained:

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \mu} &= \frac{n}{\beta} - 2 \sum_{i=1}^n \frac{\exp(-(y_i - \mu)/\beta)}{(1 + \exp(-(y_i - \mu)/\beta))\beta} + 2 \sum_{i=1}^n \frac{2\alpha b_i}{\beta(1 + b_i^2)} \\ \frac{\partial l(\theta)}{\partial \beta} &= -\frac{n}{2\beta} - \sum_{i=1}^n \frac{-(y_i - \mu)}{\beta^2} - 2 \sum_{i=1}^n \frac{(y_i - \mu) \exp(-(y_i - \mu)/\beta)}{(1 + \exp(-(y_i - \mu)/\beta))\beta^2} + 2 \sum_{i=1}^n \frac{2\alpha(y_i - \mu)b_i}{\beta^2(1 + b_i^2)} \\ \frac{\partial l(\theta)}{\partial \alpha} &= -\frac{n(80\pi^2 \alpha + 28\pi^4 \alpha^3)}{60 + 40\pi^2 \alpha^2 + 7\pi^4 \alpha^4} + 2 \sum_{i=1}^n \frac{2(y_i - \mu)b_i}{\beta(1 + b_i^2)} \end{aligned}$$

where, $b_i = \left(1 - \frac{\alpha(y_i - \mu)}{\beta}\right)$

The MLE's of the parameters of $BASLG_2(\alpha, \mu, \beta)$ distribution are estimated numerically by maximizing the likelihood function using GenSA package in R.

For the applicability of the proposed model, we analyze the model with three real life datasets obtained from different sources. We then compared our proposed distribution $BASLG_2(\alpha, \mu, \beta)$ with the normal distribution $N(\mu, \sigma^2)$, the logistic distribution $LG(\mu, \beta)$, the Laplace distribution $La(\mu, \beta)$, the skew-normal distribution $SN(\lambda, \mu, \sigma)$ of Azzalini (1985), the skew-logistic distribution $SLG(\lambda, \mu, \beta)$ of Wahed and Ali (2001), the skew-Laplace distribution $SLa(\lambda, \mu, \beta)$ of Nekoukhou and Alamatsaz (2012), the alpha-skew-normal distribution $ASN(\alpha, \mu, \sigma)$ of Elal-Olivero (2010), the alpha-skew-Laplace distribution $ASLa(\alpha, \mu, \beta)$ of Harandi and Alamatsaz (2013), the alpha-skew-logistic distribution $ASLG(\alpha, \mu, \beta)$ of Hazarika and Chakraborty (2014), the alpha-beta-skew-normal distribution $ABSN(\alpha, \beta, \mu, \sigma)$ and the beta-skew-normal distribution $BSN(\beta, \mu, \sigma)$ of Shafiei et al. (2016).

The MLE's of the parameters of different distributions mentioned above are estimated using numerical optimization schedule. AIC and BIC are used for model selection or comparison.

Example 1: N latitude degrees dataset:

This dataset is related to N latitude degrees in 69 samples from world lakes, which appear in Column 5 of the Diversity data set in website:

<http://users.stat.umn.edu/sandy/courses/8061/datasets/lakes.lsp>. Table 1 shows the MLE's, log-likelihood, AIC and BIC of the above mentioned distributions. The graphical representation of the results taking only the top three competitors for the proposed model is given in figure 9.

Table 1: MLE's, log-likelihood, AIC and BIC for N latitude degrees in 69 samples from world lakes.

Distributions	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu, \sigma^2)$	45.165	9.549	--	--	--	-253.599	511.198	515.666
$LG(\mu, \beta)$	43.639	--	--	--	4.493	-246.65	497.290	501.758
$SN(\lambda, \mu, \sigma)$	35.344	13.7	3.687	--	--	-243.04	492.072	498.774
$BSN(\beta, \mu, \sigma)$	54.47	5.52	--	--	0.74	-242.53	491.060	497.760
$SLG(\lambda, \mu, \beta)$	36.787	--	2.828	--	6.417	-239.05	490.808	490.808
$La(\mu, \beta)$	43	--	--	--	5.895	-239.25	482.496	486.964
$ASLG(\alpha, \mu, \beta)$	49.087	--	--	0.861	3.449	-237.35	480.702	487.404
$SLa(\lambda, \mu, \beta)$	42.3	--	0.255	--	5.943	-236.90	479.799	486.501
$ASLa(\alpha, \mu, \beta)$	42.3	--	--	-0.22	5.44	-236.08	478.159	484.861
$ASN(\alpha, \mu, \sigma)$	52.147	7.714	--	2.042	--	-235.37	476.739	483.441
$ABSN(\alpha, \beta, \mu, \sigma)$	53.28	9.772	--	2.943	-0.292	-234.36	476.719	485.655
$BASLG_2(\alpha, \mu, \beta)$	52.494	--	--	0.907	2.638	-230.75	467.501	474.203

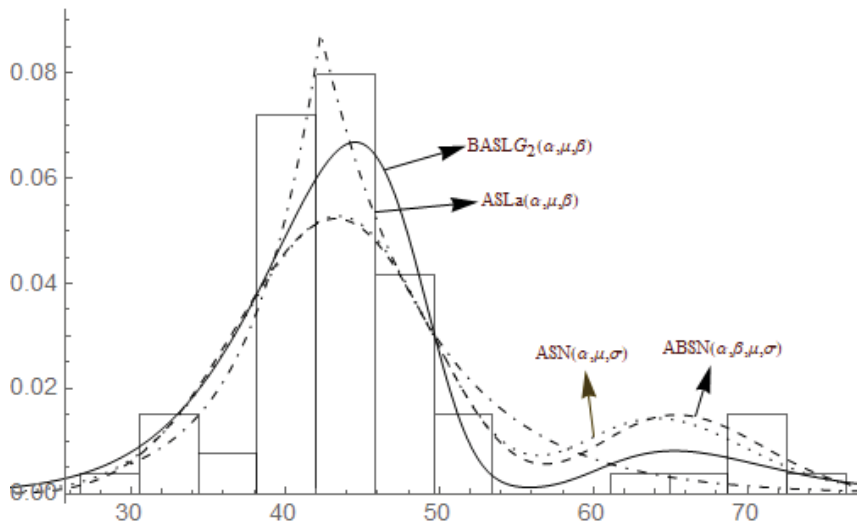


Figure 9: Plots of observed and expected densities of some distributions for N latitude degrees in 69 samples from world lakes.

Example 2: Exchange rate dataset:

To illustrate more we used the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003. The data obtained from the website <http://www.globalfindata.com>. The MLE's, log-likelihood, AIC and BIC of the above given distributions are shown in Table 2 and graphical representation are given in figure 10.

Table 2: MLE's, log-likelihood, AIC and BIC for the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003.

Distributions	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu, \sigma^2)$	4.117	1.381	--	--	--	-355.265	714.529	721.165
$SN(\lambda, \mu, \sigma)$	3.589	1.478	0.501	--	--	-355.217	716.434	726.388
$LG(\mu, \beta)$	4.251	--	--	--	0.753	-351.192	706.385	713.021
$SLG(\lambda, \mu, \beta)$	5.36	--	-2.371	--	1.018	-341.391	688.782	698.736
$La(\mu, \beta)$	4.754	--	--	--	0.971	-339.315	682.63	689.265
$BSN(\beta, \mu, \sigma)$	4.526	0.974	--	--	0.181	-334.488	674.975	684.93
$ASN(\alpha, \mu, \sigma)$	3.656	0.883	--	-3.504	--	-317.946	641.892	651.847
$SLa(\lambda, \mu, \beta)$	4.855	--	1.506	--	1	-311.318	628.636	638.59
$ABSN(\alpha, \beta, \mu, \sigma)$	3.581	1.122	--	-5.542	0.715	-301.788	611.576	624.848
$ASLG(\alpha, \mu, \beta)$	3.764	--	--	-2.025	0.403	-301.963	609.927	619.881
$ASLa(\alpha, \mu, \beta)$	4.861	--	--	0.539	0.677	-301.443	608.885	618.84
$BASLG_2(\alpha, \mu, \beta)$	3.671	--	--	-1.693	0.265	-300.583	607.166	617.121

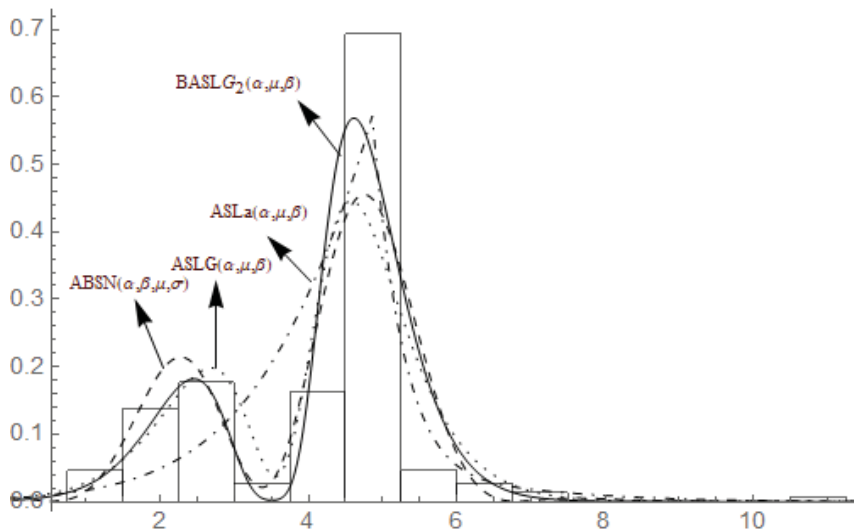


Figure 10: Plots of observed and expected densities of some distributions for the exchange rate data of the United Kingdom Pound to the United States Dollar from 1800 to 2003.

Example 3: Velocities of 82 distant galaxies dataset:

The third dataset consists of the velocities of 82 distant galaxies, diverging from our own galaxy. The data set is available at <http://www.stats.bris.ac.uk/~peter/mixdata>. The results for the above mentioned distributions are shown in Table 3 and their graphical representations are given in figure 11.

Table 3: MLE's, log-likelihood, AIC and BIC for the velocities of 82 distant galaxies, diverging from our own galaxy.

Distributions	μ	σ	λ	α	β	$\log L$	AIC	BIC
$N(\mu, \sigma^2)$	20.832	4.54	--	--	--	-240.42	484.833	489.646
$SN(\lambda, \mu, \sigma)$	24.61	5.907	-1.395	--	--	-239.21	484.42	491.64
$SLG(\lambda, \mu, \beta)$	21.532	--	-0.154	--	2.219	-233.31	472.628	479.849
$LG(\mu, \beta)$	21.075	--	--	--	2.204	-233.65	471.299	476.113
$BSN(\beta, \mu, \sigma)$	20.596	3.26	--	--	-0.158	-232.22	470.44	477.66
$ASN(\alpha, \mu, \sigma)$	17.417	3.869	--	-1.656	--	-230.09	466.175	473.395
$SLa(\lambda, \mu, \beta)$	20.846	--	1.002	--	2.997	-228.83	463.658	470.878
$La(\mu, \beta)$	20.838	--	--	--	2.997	-228.83	461.66	466.474
$ASLG(\alpha, \mu, \beta)$	18.482	--	--	-0.833	1.646	-224.88	455.754	462.974
$ABSN(\alpha, \beta, \mu, \sigma)$	19.448	3.462	--	-1.392	0.323	-220.06	448.109	457.736
$ASLa(\alpha, \mu, \beta)$	19.473	--	--	-0.842	1.805	-220.79	447.586	454.806
$BASLG_2(\alpha, \mu, \beta)$	17.117	--	--	-0.799	1.263	-219.86	445.716	452.936

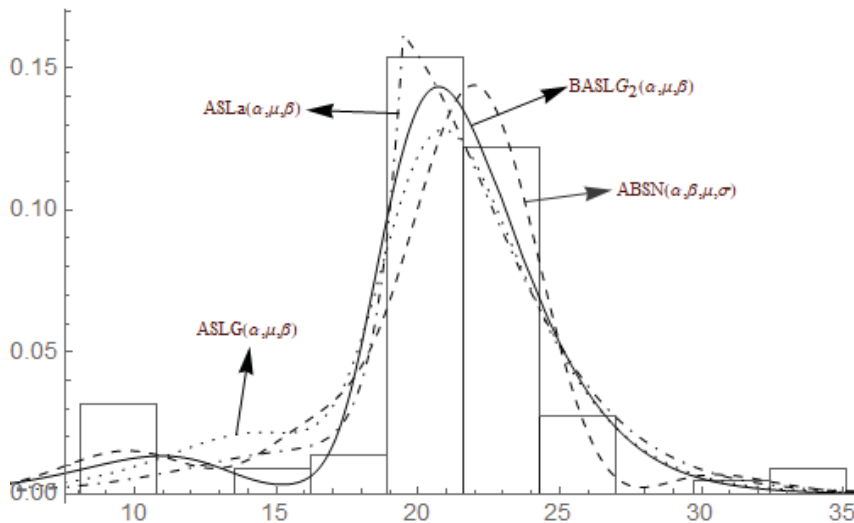


Figure 11: Plots of observed and expected densities of some distributions for the velocities of 82 distant galaxies, diverging from our own galaxy.

From the above table 1 - 3 it is observed that based on AIC and BIC criteria our proposed distribution provides better fitting among others known related distributions. The figures 9 – 11, which is the plot of observed and expected densities, also confirm our findings.

4.3. Likelihood Ratio Test

The likelihood ratio (LR) test is used to discriminate between $LG(\mu, \beta)$ and $BASLG_2(\alpha, \mu, \beta)$ as they are nested models. Here the Null hypothesis $H_0: \alpha = 0$, that is the sample is drawn from $LG(\mu, \beta)$; against the alternative $H_1: \alpha \neq 0$, that is the sample is drawn from $BASLG_2(\alpha, \mu, \beta)$.

The values of the LR test statistic for the above three examples are given in Table 4.

Table 4: the values of the LR test statistic for different datasets

Datasets	Example 1	Example 2	Example 3
LR test statistic value	31.7894	101.2184	27.5838

Since the calculated value of LR test statistic for all the datasets are greater than 6.635, the 99% tabulated value of chi-square at 1 degree of freedom; therefore there is no evidence in support of the null hypothesis for all the datasets. Thus, we conclude that for all the examples the sampled data comes from $BASLG_2(\alpha, \mu, \beta)$, not from $LG(\mu, \beta)$.

5. CONCLUSIONS

There are many real life situations where the asymmetry and bimodality behaviours are observed e.g., the time between eruptions of certain geysers, the size of worker weaver ants, the age of incidence of certain diseases; in age distribution and growth estimates fish population, in geological studies distribution of the sediments etc. In this paper a new bimodal alpha-skew-logistic distribution

which includes both unimodal and bimodal behavior is introduced and studied some of its basic properties. Here, three real life data sets are considered for data modelling to show the usefulness of the proposed distribution. Finally, based on AIC and BIC it is observed that our proposed distribution provides better fitting comparing to the other known distributions.

6. REFERENCES

- Azzalini, A., 1985, A class of distributions which includes the normal ones. *Scandinavian journal of statistics*, **12(2)**, 171-178.
- Bahrami, W., Agahi, H., Rangin, H., 2009, A two-parameter Balakrishnan skew-normal distribution. *Journal of Statistical Research of Iran*, **6(2)**, 231-242.
- Balakrishnan, N., 1992, Handbook of Logistic Distribution. Marcel Dekker. New York.
- Balakrishnan, N., 2002, Discussion on "Skew multivariate models related to hidden truncation and/or selective reporting" by B.C. Arnold and R.J. Beaver. *Test* **11**, 37-39
- Chakraborty, S., Hazarika, P.J., Ali, M.M., 2012, A New Skew Logistic Distribution and its Properties. *Pakistan Journal of Statistics*, **28(4)**, 513-524.
- Chakraborty, S., and Hazarika, P.J., 2011, A survey of the theoretical developments in univariate skew normal distributions. *Assam Statistical Review*, **25(1)**, 41-63.
- Chakraborty, S., Hazarika, P.J., Ali, M.M., 2015, A multimodal skewed extension of normal distribution: its properties and applications. *Statistics*, **49(4)**, 859-877.
- Dimitrov, B., Green, Jr.D., Chukova, S., 1997, Probability distributions in periodic random environment and their applications. *SIAM Journal on Applied Mathematics*, **57(2)**, 501-517.
- Elal-Olivero, D., 2010, Alpha-skew-normal distribution. *Proyecciones (Antofagasta)*, **29(3)**, 224-240.
- Gumbel, E.J., 1961, Bivariate logistic distributions. *Journal of the American Statistical Association*, **56(294)**, 335-349.
- Hammel, I., Lagunoff, D., Bauza, M., Chi, E., 1983, Periodic, multimodal distribution of granule volumes in mast cells. *Cell and tissue research*, **228(1)**, 51-59.
- Harandi, S.S., Alamatsaz, M.H., 2013, Alpha-Skew-Laplace distribution. *Statistics and Probability Letters*, **83(3)**, 774-782.
- Hasanalipour P, Sharafi M (2012) A new generalized Balakrishnan skew-normal distribution. *Statistical papers*, **53(1)**, 219-228.
- Hazarika, P.J., Chakraborty. S., 2014, Alpha-Skew-Logistic Distribution. *IOSR Journal of Mathematics*, **10(4)**, 36-46.
- Hazarika, P.J., Chakraborty, S., 2015, A note on alpha-skew-generalized logistic distribution. *IOSR Journal of Mathematics*, **11(1)**, 85-88.
- Huang, W.J., Chen, Y.H., 2007, Generalized skew-Cauchy distribution. *Statistics and Probability Letters*, **77(11)**, 1137-1147.
- Louzada, F., Ara, A., Fernandes, G., 2017, The bivariate alpha-skew-normal distribution. *Communications in Statistics-Theory and Methods*, **46(14)**, 7147-7156.

Nekoukhou, V., Alamatsaz, M.H., 2012, A family of skew-symmetric-Laplace distributions. *Statistical Papers*, **53**(3), 685-696.

Prudnikov, A.P., Brychkov, Y.A., Marichev, Y.I., 1986, Integrals and Series. (Volume-I), Gordon and Breach, Amsterdam.

Shafiei, S., Doostparast, M., Jamalizadeh, A., 2016, The alpha-beta skew normal distribution: Properties and applications. *Statistics*, **50**(2), 338-349.

Sharafi, M., Behboodian, J., 2008, The Balakrishnan skew-normal density. *Statistical Papers*, **49**(4), 769-778.

Sharafi, M., Sajjadnia, Z., Behboodian, J., 2017, A new generalization of alpha-skew-normal distribution. *Communications in Statistics-Theory and Methods*, **46**(12), 6098-6111.

Sinha, R.K., 2012, A thought on exotic statistical distributions. *World Academy of Science, Engineering and Technology*, **61**, 366-369.

Venegas, O., Bolfarine, H., Gallardo, D.I., Vergara-Fernández, A., Gómez, H.W., 2016, A Note on the Log-Alpha-Skew-Normal Model with Geochemical Applications. *Appl. Math.*, **10**(5), 1697-1703.

Wahed, A., Ali, M.M. 2001, The skew-logistic distribution. *J. Statist. Res.*, **35**(2), 71-80.

Yadegari, I., Gerami, A., Khaledi, M.J., 2008, A generalization of the Balakrishnan skew normal distribution. *Stat. Probab. Lett.*, **78**, 1165-1167.

APPENDIX

A: Proof of Theorem 1

$$\begin{aligned}
 F_Z(z; \alpha) &= \frac{1}{C_2(\alpha)} \int_{-\infty}^z [(1 - \alpha z)^2 + 1]^2 \frac{e^{-z}}{(1 + e^{-z})^2} dz \\
 &= \frac{1}{C_2(\alpha)} \left[\int_{-\infty}^z \frac{4e^{-z}}{(1 + e^{-z})^2} dz - 8\alpha \int_{-\infty}^z \frac{ze^{-z}}{(1 + e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^z \frac{z^2 e^{-z}}{(1 + e^{-z})^2} dz - 4\alpha^3 \int_{-\infty}^z \frac{z^3 e^{-z}}{(1 + e^{-z})^2} dz + \alpha^4 \int_{-\infty}^z \frac{z^4 e^{-z}}{(1 + e^{-z})^2} dz \right] \\
 &= \frac{1}{C_2(\alpha)} \left[\frac{4e^z}{1 + e^z} - \frac{8\alpha z e^z}{1 + e^z} + 8\alpha^2 z \left(\frac{ze^z}{1 + e^z} - 2 \operatorname{Log}[1 + e^z] \right) - 2 \operatorname{Li}_2[-e^z] \right] - \\
 &\quad 4\alpha^3 \left(z^2 \left(z - \frac{z}{1 + e^z} - 3 \operatorname{Log}[1 + e^z] \right) - 6z \operatorname{Li}_2[-e^z] + 6 \operatorname{Li}_3[-e^z] \right) + \\
 &\quad \alpha^4 \left(\frac{e^z z^4}{1 + e^z} - 4z^3 \operatorname{Log}[1 + e^z] - 12z^2 \operatorname{Li}_2[-e^z] + 24z \operatorname{Li}_3[-e^z] - 24 \operatorname{Li}_4[-e^z] \right) \Big]
 \end{aligned}$$

On Simplification we get the result in equation (8).

B: Proof of Theorem 3

$$M_Z(t) = \frac{1}{C_2(\alpha)} \int_{-\infty}^{\infty} e^{tz} \frac{[(1 - \alpha z)^2 + 1]^2 e^{-z}}{(1 + e^{-z})^2} dz$$

$$\begin{aligned}
 &= \frac{1}{C_2(\alpha)} \left[\int_{-\infty}^{\infty} \frac{4e^{z(t-1)}}{(1+e^{-z})^2} dz - 8\alpha \int_{-\infty}^{\infty} \frac{ze^{z(t-1)}}{(1+e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^{\infty} \frac{z^2e^{z(t-1)}}{(1+e^{-z})^2} dz - 4\alpha^3 \int_{-\infty}^{\infty} \frac{z^3e^{z(t-1)}}{(1+e^{-z})^2} dz + \alpha^4 \int_{-\infty}^{\infty} \frac{z^4e^{z(t-1)}}{(1+e^{-z})^2} dz \right] \\
 &= \frac{1}{C_2(\alpha)} \left[4(\pi \operatorname{Csc}(\pi)) - 8\alpha(\pi(1-\pi \operatorname{Cot}(\pi))\operatorname{Csc}(\pi)) + 8\alpha^2 \left(\frac{1}{2} \pi^2 \operatorname{Csc}(\pi)^3 (\pi(3+\operatorname{Cos}(2\pi)) - \right. \right. \\
 &\quad \left. \left. 2\operatorname{Sin}(2\pi)) \right) - 4\alpha^3 \left(-\frac{1}{4} \pi^3 \operatorname{Csc}(\pi)^4 (\pi(23\operatorname{Cos}(\pi) + \operatorname{Cos}(3\pi)) - 3(5\operatorname{Sin}(\pi) + \operatorname{Sin}(3\pi))) \right) + \right. \\
 &\quad \left. \alpha^4 \left(\frac{1}{8} \pi^4 \operatorname{Csc}(\pi)^5 (115\pi + \pi(76\operatorname{Cos}(2\pi) + \operatorname{Cos}(4\pi)) - 88\operatorname{Sin}(2\pi) + \operatorname{Sin}(4\pi)) \right) \right]
 \end{aligned}$$

On simplification we get the desired result in equation (12).

C: Proof of Theorem 4

Since, the k^{th} moment of the standard logistic distribution (see Balakrishnan (1992)) is given by

$$E(X^k) = \begin{cases} 2 \left(1 - \frac{1}{2^{k-1}} \right) \Gamma(k+1) \zeta(k) ; \text{ when } k \text{ is even} \\ 0 ; \text{ when } k \text{ is odd} \end{cases} \quad (\text{A1})$$

$$\begin{aligned}
 \text{Now, } E(Z^k) &= \frac{1}{C_2(\alpha)} \int_{-\infty}^{\infty} Z^k \frac{[(1-\alpha z)^2 + 1]^2 e^{-z}}{(1+e^{-z})^2} dz \\
 &= \frac{1}{C_2(\alpha)} \left[\int_{-\infty}^{\infty} \frac{4z^k e^{-z}}{(1+e^{-z})^2} dz - 8\alpha \int_{-\infty}^{\infty} \frac{z^{k+1} e^{-z}}{(1+e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^{\infty} \frac{z^{k+2} e^{-z}}{(1+e^{-z})^2} dz - 4\alpha^3 \int_{-\infty}^{\infty} \frac{z^{k+3} e^{-z}}{(1+e^{-z})^2} dz + \alpha^4 \int_{-\infty}^{\infty} \frac{z^{k+4} e^{-z}}{(1+e^{-z})^2} dz \right]
 \end{aligned}$$

Case1: Let k is even, then we get

$$\begin{aligned}
 E(Z^k) &= \frac{1}{C_2(\alpha)} \left(\int_{-\infty}^{\infty} \frac{4z^k e^{-z}}{(1+e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^{\infty} \frac{z^{k+2} e^{-z}}{(1+e^{-z})^2} dz + \alpha^4 \int_{-\infty}^{\infty} \frac{z^{k+4} e^{-z}}{(1+e^{-z})^2} dz \right) \\
 &= \frac{1}{C_2(\alpha)} \left[4 \left(1 - \frac{1}{2^{k-1}} \right) \Gamma(k+1) \zeta(k) + 16\alpha^2 \left(1 - \frac{1}{2^{k+1}} \right) \Gamma(k+3) \zeta(k+2) + 2\alpha^4 \left(1 - \frac{1}{2^{k+3}} \right) \Gamma(k+5) \zeta(k+4) \right]
 \end{aligned}$$

[Using eqn. (A1)]

On simplification we get the result in equation (14).

Case2: Let k is odd, then we get

$$\begin{aligned}
 E(Z^k) &= \frac{1}{C_2(\alpha)} \left(-8\alpha \int_{-\infty}^{\infty} \frac{z^{k+1} e^{-z}}{(1+e^{-z})^2} dz - 4\alpha^3 \int_{-\infty}^{\infty} \frac{z^{k+3} e^{-z}}{(1+e^{-z})^2} dz \right) \\
 &= \frac{1}{C_2(\alpha)} \left[-16\alpha \left(1 - \frac{1}{2^k} \right) \Gamma(k+2) \zeta(k+1) - 8\alpha^3 \left(1 - \frac{1}{2^{k+2}} \right) \Gamma(k+5) \zeta(k+3) \right] \quad [\text{Using eqn. (A1)}]
 \end{aligned}$$

On simplification we get the result (15).

D: Normalizing Constants

$$C_2(\alpha, \alpha_1, \alpha_2) = \frac{1}{15} \left(60 + 7\pi^4 \alpha_1^4 + 60\pi^2 \alpha_1^3 \alpha_2 + 40\pi^2 \alpha_2^2 + 7\pi^4 \alpha_2^4 + 10\pi^2 \alpha_1^2 (4 + \pi^2 \alpha_2^2) + \right.$$

$$\left. \Psi(z_1, z_2, \alpha) = \frac{e^{-z_1 - z_2}}{(1+e^{-z_1})^2 (1+e^{-z_2})^2} \left[1 + \alpha \left(\frac{1-e^{-z_1}}{1+e^{-z_1}} \right) \left(\frac{1-e^{-z_2}}{1+e^{-z_2}} \right) \right] \right)$$

$$C_2(\alpha_1, \alpha_2) = \frac{1}{105} \left(1680 + \pi^2 (224\alpha_1\alpha_2(10 + 7\pi^2\alpha_2^2) + 28\alpha_2^2(40 + 7\pi^2\alpha_2^2) + 16\pi^2\alpha_1^3\alpha_2(98 + 155\pi^2\alpha_2^2) + 8\alpha_1^2(140 + 392\pi^2\alpha_2^2 + 155\pi^4\alpha_2^4) + \pi^2\alpha_1^4(196 + 1240\pi^2\alpha_2^2 + 889\pi^4\alpha_2^4)) \right)$$

$$C_2(\alpha, \beta) = \frac{1}{15015} \left(60060 + 40040\pi^2\alpha^2 + 7007\pi^4\alpha^4 + 11212\pi^4\alpha\beta + 88660\pi^6\alpha^3\beta + 177320\pi^6\beta^2 + 762762\pi^8\alpha^2\beta^2 + 465010\pi^{10}\alpha\beta^3 + 15559247\pi^{12}\beta^4 \right)$$

E: Skewness and Kurtosis

$$\beta_1 = \frac{11200\pi^2\alpha^2(10080 + 20240\pi^2\alpha^2 + 6320\pi^4\alpha^4 - 5112\pi^6\alpha^6 - 5194\pi^8\alpha^8 - 1519\pi^{10}\alpha^{10})^2}{(8400 + 17920\pi^2\alpha^2 + 10280\pi^4\alpha^4 + 3456\pi^6\alpha^6 + 1085\pi^8\alpha^8)^3}$$

$$\beta_2 = \frac{\left(7(42336000 + 207360000\pi^2\alpha^2 + 321427200\pi^4\alpha^4 + 306867200\pi^6\alpha^6 + 231988640\pi^8\alpha^8 + 126595840\pi^{10}\alpha^{10} + 42918512\pi^{12}\alpha^{12} + 7353920\pi^{14}\alpha^{14} + 304927\pi^{16}\alpha^{16}) \right)}{(8400 + 17920\pi^2\alpha^2 + 10280\pi^4\alpha^4 + 3456\pi^6\alpha^6 + 1085\pi^8\alpha^8)^2}$$

F: Proof of Theorem 5

$$F_1(z; \alpha) = \frac{1}{C_2(\alpha)} \int_{-\infty}^z \frac{(4 + 8\alpha^2 z^2 + \alpha^4 z^4)e^{-z}}{(1 + e^{-z})^2} dz$$

$$= \frac{1}{C_2(\alpha)} \left[\int_{-\infty}^z \frac{4e^{-z}}{(1 + e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^z \frac{z^2 e^{-z}}{(1 + e^{-z})^2} dz + \alpha^4 \int_{-\infty}^z \frac{z^4 e^{-z}}{(1 + e^{-z})^2} dz \right]$$

$$= \frac{1}{C_2(\alpha)} \left[\frac{4e^z}{1 + e^z} + 8\alpha^2 z \left(\left(\frac{ze^z}{1 + e^z} - 2 \operatorname{Log}[1 + e^z] \right) - 2 \operatorname{Li}_2[-e^z] \right) + \right.$$

$$\left. \alpha^4 \left(\frac{e^z z^4}{1 + e^z} - 4z^3 \operatorname{Log}[1 + e^z] - 12z^2 \operatorname{Li}_2[-e^z] + 24z \operatorname{Li}_3[-e^z] - 24 \operatorname{Li}_4[-e^z] \right) \right]$$

On simplification we get the result in equation (18).

G: Proof of Theorem 6

$$M_Z(t; \alpha) = \frac{1}{C_2(\alpha)} \int_{-\infty}^{\infty} e^{tz} \frac{(4 + 8\alpha^2 z^2 + \alpha^4 z^4)e^{-z}}{(1 + e^{-z})^2} dz$$

$$= \frac{1}{C_2(\alpha)} \left[\int_{-\infty}^{\infty} \frac{4e^{z(t-1)}}{(1 + e^{-z})^2} dz + 8\alpha^2 \int_{-\infty}^{\infty} \frac{z^2 e^{z(t-1)}}{(1 + e^{-z})^2} dz + \alpha^4 \int_{-\infty}^{\infty} \frac{z^4 e^{z(t-1)}}{(1 + e^{-z})^2} dz \right]$$

$$= \frac{1}{C_2(\alpha)} \left[4(\pi \operatorname{Csc}[\pi]) + 8\alpha^2 \left(\frac{1}{2} \pi^2 \operatorname{Csc}[\pi]^3 (\pi(3 + \operatorname{Cos}[2\pi]) - 2 \operatorname{Sin}[2\pi]) \right) + \right.$$

$$\left. \alpha^4 \left(\frac{1}{8} \pi^4 \operatorname{Csc}[\pi]^5 (115\pi + \pi(76 \operatorname{Cos}[2\pi] + \operatorname{Cos}[4\pi]) - 88 \operatorname{Sin}[2\pi] + \operatorname{Sin}[4\pi]) \right) \right]$$

On simplification we get the result in equation (19).