

Conservation Laws of Some Fractional Nonlinear Evolution Equations

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ABSTRACT

In the present work, we formulate the geometrical interpretation of the conformable fractional pseudo-spherical surfaces. We derive the relationship between the fractional quasi AKNS-like system and conformable fractional pseudo-spherical surfaces. Besides, we construct the fractional quasi conservation laws for some fractional equations from the defined fractional quasi AKNS-like system and the fractional matrix one-form. Some nonlinear fractional differential equations have been studied; certainly are the space-time fractional versions of the Liouville equation, the sine-Gordon equation, the sinh-Gordon equation, the cos-Gordon equation, the cosh-Gordon equation, and the nonlinear Schrödinger equation.

Keywords: Fractional evolution equation; fractional quasi AKNS method; fractional quasi conservation laws; conformable fractional derivative.

Mathematics Subject Classification: 26A33, 34A08, 35R11

1. INTRODUCTION

The mutual relations of the partial differential equations with the geometry of surfaces, or submanifolds three dimensional space, have an important role in mathematics and physics ,Loutsenko and Roubtsov (1997), Tajiri and Maesono (1997), Albowitz and Clarkson (1991), Ghosh et al. (2016). Examples are; the relationship between surfaces of a constant negative Gaussian curvature in Euclidean three-space, the Sine-Gordon Equation and Bäcklund transformations which are relevant to the given equation and the original Bäcklund transformation for the Sine-Gordon Equation, Kivshar and Melomend (1989), Ablowitz and Segur (1981).

The relation between the class of nonlinear partial differential equations and pseudo-spherical surfaces (pss) was observed by Sasaki (1983); where the solution of these class of equations is

performed using Ablowitz, Kaup, Newell, and Segur (AKNS) 2×2 inverse scattering method (ISM), Sakovich and Sakovich (2005), Khater et al. (1997), Khater et al. (1999), Khater et al. (2004). Although, fractional calculus is an old topic; Leibniz; it has attracted great attention from mathematicians, physicists, and engineers in recent decades, Podlubny (1999), Kilbas et al. (2006), Abdel-Salam and Mourad (2019).

There are many types of fractional derivatives, i.e. Riemann–Liouville, Caputo, Grunwald–Letnikov, Riesz and Wey. When comparing these definitions with Newton's derivative, some inconsistencies arise. Examples of this inconsistencies are; fractional derivatives do not obey product rule, quotient rule, chain rule for derivative operations, Khalil et al. (2014), Abdeljawad et al. (2017), Jarad et al. (2017), Abdeljawad et al. (2019).

One possible concept of local fractional derivative (LFD) is introduced by Khalil et al. (2014), called conformable fractional derivative (CFD). The properties of CFD are exactly similar to Newton derivative and can be easily used to solve fractional differential equations, Khalil et al. (2014).

The CFD is defined as

$$D^\alpha f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \quad \forall t > 0, \alpha \in (0, 1], \quad (1)$$

$$f^{(\alpha)}(0) = \lim_{t \rightarrow 0^+} f^{(\alpha)}(t). \quad (2)$$

Here $f^{(\alpha)}(0)$ is not defined. When $\alpha = 1$, this fractional derivative reduces to the ordinary derivative. The CFD has the following properties:

$$D^\alpha t^p = p t^{p-\alpha}, \quad p \in \mathbb{R}, \quad D^\alpha c = 0, \quad \forall f(t) = c, \quad (3)$$

$$D^\alpha (a f + b g) = a D^\alpha f + b D^\alpha g, \quad \forall a, b \in \mathbb{R}, \quad (4)$$

$$D^\alpha (f g) = f D^\alpha g + g D^\alpha f, \quad (5)$$

$$D^\alpha f(g) = \frac{df}{dg} D^\alpha g, \quad D^\alpha f(t) = t^{1-\alpha} \frac{df}{dg}, \quad (6)$$

where f, g are two α -differentiable functions and c is an arbitrary constant, Khalil et al. (2014). The CFD of some functions are

$$\begin{aligned} D^\alpha e^{ct} &= c t^{1-\alpha} e^{ct}, \quad D^\alpha \sin(ct) = c t^{1-\alpha} \cos(ct), \quad D^\alpha \cos(ct) = -c t^{1-\alpha} \sin(ct), \\ D^\alpha e^{c t^\alpha} &= c \alpha e^{c t^\alpha}, \quad D^\alpha \sin(c t^\alpha) = c \alpha \cos(c t^\alpha), \quad D^\alpha \cos(c t^\alpha) = -c \alpha \sin(c t^\alpha). \end{aligned} \quad (7)$$

In the following sections, we apply this definition of derivative to pseudo-spherical surfaces and AKNS technique.

In this paper, we discuss the geometrical interpretation of special properties (such as an infinite number of quasi conservation laws and Bäcklund transformations (BTs)) for solutions of the fractional differential equation which describe fractional pss.

2. GEOMETRICAL INTERPRETATION OF THE FRACTIONAL PSEUDO-SPHERICAL SURFACES

The linear system of conformable fractional differential equations can be defined as $D_x^\alpha v = Av$, $D_t^\alpha v = Bv$, and the fractional integrability condition $D_t^\alpha(D_x^\alpha v) = D_x^\alpha(D_t^\alpha v)$, gives

$D_t^\alpha A - D_x^\alpha B + [A, B] = 0$, where A, B are 2×2 matrix depending on λ, u and its fractional derivatives.

Let g_α be a fractional Riemannian metric on M^2 , ∇^α the corresponding Levi-Civita connection on the conformable fractional tangent bundle $cfTM^2$, $\{e_{1_\alpha}, e_{2_\alpha}\}$ be a moving orthonormal fractional frame on some open domain $U \subset M^2$ and $\{\omega_{1_\alpha}, \omega_{2_\alpha}\}$ a corresponding moving fractional coframe. The relations $\nabla^\alpha e_{i_\alpha} = \omega_{i_\alpha}^{j_\alpha} \otimes e_{j_\alpha}$ define the connection of the fractional one-form matrix with respect to the fractional frame $\{e_{1_\alpha}, e_{2_\alpha}\}$. The orthogonality of this fractional frame implies that $\omega_{1_\alpha}^{1_\alpha} = \omega_{2_\alpha}^{2_\alpha} = 0$, $\omega_{1_\alpha}^{2_\alpha} = -\omega_{2_\alpha}^{1_\alpha} = \omega_{3_\alpha}$. Hence the fractional Levi-Civita connection one-form on the fractional tangent bundle fTM^2 with respect to the fractional moving frame $\{e_{1_\alpha}, e_{2_\alpha}\}$ is

$$\begin{pmatrix} 0 & \omega_{3_\alpha} \\ -\omega_{3_\alpha} & 0 \end{pmatrix} \quad (8)$$

It yields the following structural equations:

$$d^\alpha \omega_{1_\alpha} = \omega_{3_\alpha} \wedge \omega_{2_\alpha}, \quad d^\alpha \omega_{2_\alpha} = \omega_{1_\alpha} \wedge \omega_{3_\alpha}. \quad (9)$$

The fractional Gaussian curvature κ_α of the space M^2 is defined by the fractional Gauss equation

$$d^\alpha \omega_{3_\alpha} = \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha}. \quad (10)$$

The formula for some local connections on a fractional two-dimensional real Riemannian manifold M^2 ; which is quite relevant in the theory of nonlinear fractional partial differential equations; could be written as

$$\Omega_\alpha = \frac{1}{2} \begin{pmatrix} \omega_{2_\alpha} & \omega_{1_\alpha} - \omega_{3_\alpha} \\ \omega_{1_\alpha} + \omega_{3_\alpha} & -\omega_{2_\alpha} \end{pmatrix}, \quad (11)$$

as a new connection form for some (non-specified) fractional bundles over M^2 . The key property of the fractional matrix one-form Ω_α is that it satisfies the fractional curvature condition

$$\Theta_\alpha \equiv d^\alpha \Omega_\alpha - \Omega_\alpha \wedge \Omega_\alpha = 0, \quad (12)$$

if $\kappa_\alpha = -1$ on U . In the following section, we give some examples where different fractional matrix one-forms were discussed, depending on a function u (or some functions) of some

independent variables, such that the fractional curvature condition $\Theta_\alpha = 0$ for this form is equivalent to one of the well known fractional differential equations having an infinite number of conservation laws. If we put $\alpha = 1$ in equations (8- 12) will be reduced to equations (1- 6) in Khater et al. (1997). We know that The moving frame and the associated linear scattering problems are defined for the entire real t and x , that is $-\infty < t < \infty, -\infty < x < \infty$. The term quasi due to the fractional variables are must be positive, that is $0 < t < \infty, 0 < x < \infty$.

The fractional matrix one-form Ω_α can be written as

$$\Omega_\alpha = \frac{1}{2} \begin{pmatrix} \omega_{2_\alpha} & \omega_{1_\alpha} - \omega_{3_\alpha} \\ \omega_{1_\alpha} + \omega_{3_\alpha} & -\omega_{2_\alpha} \end{pmatrix} = \begin{pmatrix} \frac{\eta}{2} d^\alpha x + A d^\alpha t & q d^\alpha x + B d^\alpha t \\ r d^\alpha x + C d^\alpha t & -\frac{\eta}{2} d^\alpha x - A d^\alpha t \end{pmatrix} = U d^\alpha x + V d^\alpha t, \quad (13)$$

since $d^\alpha \Psi = \Omega_\alpha \Psi$, where $\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$, this leads to

$$D_x^\alpha \Psi = U \Psi, \quad U = \begin{pmatrix} \frac{\eta}{2} & q(x, t) \\ r(x, t) & -\frac{\eta}{2} \end{pmatrix}, \quad (14)$$

and

$$D_t^\alpha \Psi = V \Psi, \quad V = \begin{pmatrix} A(x, t) & B(x, t) \\ C(x, t) & -A(x, t) \end{pmatrix}, \quad (15)$$

The compatibility condition of expressions (14) and (15) which is given by cross differentiation $D_t^\alpha (D_x^\alpha \psi_i) = D_x^\alpha (D_t^\alpha \psi_i)$, ($i = 1, 2$) gives the following equation

$$D_t^\alpha U - D_x^\alpha V + [U, V] = 0, \quad (16)$$

where $[U, V] = UV - VU$, is the Lie bracket, which defines the fractional zero curvature condition. Equation (16) can be rewritten in a simple form as

$$\begin{aligned} D_x^\alpha A &= q C - r B, \\ D_x^\alpha B - \eta B &= D_t^\alpha q - 2 A q, \\ D_x^\alpha C + \eta C &= D_t^\alpha r + 2 A r. \end{aligned} \quad (17)$$

The condition $\Theta_\alpha = 0$ depends only on relation (12) and the commutative relations in the algebra $SL_\alpha(2, R)$. The fractional matrix one-form Ω_α may be written as:

$$\Omega_\alpha = \begin{pmatrix} 0 & \omega_{3_\alpha} & \omega_{1_\alpha} \\ -\omega_{3_\alpha} & 0 & \omega_{2_\alpha} \\ -\omega_{1_\alpha} & -\omega_{2_\alpha} & 0 \end{pmatrix}, \quad (18)$$

which contains the fractional Levi-Civita connection form $\begin{pmatrix} 0 & \omega_{3_\alpha} \\ -\omega_{3_\alpha} & 0 \end{pmatrix}$. In the following, we show

that there are many fractional differential equations describe conformable fractional pseudo-spherical surfaces. Examples include the space-time fractional Liouville's, sin-Gordon, sinh-Gordon, cos-Gordon, cosh-Gordon, and nonlinear Schrödinger equation. The latter equations proved to be of great importance in many physical applications, Sakovich and Sakovich (2005), Khater et al (1997), Khater et al. (2004).

Note that:

1- the total fractional derivative of the function of two variables $u(x, t)$ is

$$d^\alpha u(x, t) = \frac{d^\alpha u}{dt^\alpha} dt^\alpha + \frac{d^\alpha u}{dx^\alpha} dx^\alpha \text{ or simply } d^\alpha u = D_x^\alpha u dx^\alpha + D_t^\alpha u dt^\alpha. \text{ Also, the total fractional}$$

derivative of the first α -fractional partial derivative with respect to x ($D_x^\alpha u$) is

$$d^\alpha (D_x^\alpha u) = D_x^\alpha D_x^\alpha u dx^\alpha + D_t^\alpha D_x^\alpha u dt^\alpha.$$

2- the second exterior fractional derivative finishes ($d^\alpha(dx^\alpha) = 0$).

3- $dx^\alpha \wedge dx^\alpha = 0$, and $dt^\alpha \wedge dt^\alpha = 0$.

The procedure is clarified in the following examples.

Example 1: space-time fractional Liouville's equation.

Consider

$$\omega_{1_\alpha} = D_x^\alpha dx^\alpha, \quad \omega_{2_\alpha} = \lambda dx^\alpha + \frac{e^u}{\lambda} dt^\alpha, \quad \omega_{3_\alpha} = \frac{e^u}{\lambda} dt^\alpha, \quad (19)$$

Then M^2 is a conformable fractional pseudo-spherical surface if u satisfies the space-time fractional Liouville's equation

$$D_t^\alpha D_x^\alpha u - e^u = 0. \quad (20)$$

Proof:

$$\begin{aligned} d^\alpha \omega_{1_\alpha} &= d^\alpha (D_x^\alpha u) \wedge dx^\alpha + D_x^\alpha u \wedge d^\alpha(dx^\alpha) = (D_x^\alpha D_x^\alpha u dx^\alpha + D_t^\alpha D_x^\alpha u dt^\alpha) \wedge dx^\alpha \\ &= D_t^\alpha D_x^\alpha u dt^\alpha \wedge dx^\alpha, \end{aligned}$$

$$\omega_{3_\alpha} \wedge \omega_{2_\alpha} = \frac{e^u}{\lambda} dt^\alpha \wedge (\lambda dx^\alpha + \frac{e^u}{\lambda} dt^\alpha) = e^u dt^\alpha \wedge dx^\alpha$$

$$\text{then } d^\alpha \omega_{1_\alpha} - \omega_{3_\alpha} \wedge \omega_{2_\alpha} = (D_t^\alpha D_x^\alpha u - e^u) dt^\alpha \wedge dx^\alpha = 0, \text{ iff } D_t^\alpha D_x^\alpha u - e^u = 0,$$

$$d^\alpha \omega_{2_\alpha} = \lambda d^\alpha (dx^\alpha) + \frac{e^u}{\lambda} (D_x^\alpha u dx^\alpha + D_t^\alpha u dt^\alpha) \wedge dt^\alpha = \frac{e^u}{\lambda} D_x^\alpha u dx^\alpha \wedge dt^\alpha$$

$$\omega_{1_\alpha} \wedge \omega_{3_\alpha} = \frac{e^u}{\lambda} D_x^\alpha u dx^\alpha \wedge dt^\alpha, \quad \text{this gives } d^\alpha \omega_{2_\alpha} = \omega_{1_\alpha} \wedge \omega_{3_\alpha},$$

$$d^\alpha \omega_{3_\alpha} = \frac{e^u}{\lambda} (D_x^\alpha u dx^\alpha + D_t^\alpha u dt^\alpha) \wedge dt^\alpha = \frac{e^u}{\lambda} D_x^\alpha u dx^\alpha \wedge dt^\alpha$$

$$\kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} = \kappa_\alpha (\lambda dx^\alpha + \frac{e^u}{\lambda} dt^\alpha) \wedge D_x^\alpha u dx^\alpha = \frac{e^u \kappa_\alpha}{\lambda} D_x^\alpha u dt^\alpha \wedge dx^\alpha = -\frac{e^u \kappa_\alpha}{\lambda} D_x^\alpha u dx^\alpha \wedge dt^\alpha$$

$$\text{then } d^\alpha \omega_{3_\alpha} = \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha}, \quad \text{iff } \kappa_\alpha = -1.$$

When $\alpha = 1$, equation (19) is the same equation (15) as in (Khater et al., 1997).

Example 2: the space-time fractional sine–Gordon equation.

Consider

$$\omega_{1_\alpha} = D_x^\alpha u dx^\alpha, \quad \omega_{2_\alpha} = \lambda dx^\alpha + \frac{\cos u}{\lambda} dt^\alpha, \quad \omega_{3_\alpha} = -\frac{\sin u}{\lambda} dt^\alpha, \quad (21)$$

Then M^2 is a conformable fractional pseudo-spherical surface if u satisfies the space-time fractional sine–Gordon equation

$$D_t^\alpha D_x^\alpha u + \sin u = 0. \quad (22)$$

Proof:

$$d^\alpha \omega_{1_\alpha} = D_t^\alpha D_x^\alpha u dt^\alpha \wedge dx^\alpha, \quad \omega_{3_\alpha} \wedge \omega_{2_\alpha} = -\frac{\sin u}{\lambda} dt^\alpha \wedge (\lambda dx^\alpha + \frac{\cos u}{\lambda} dt^\alpha) = -\sin u dt^\alpha \wedge dx^\alpha$$

$$\text{then } d^\alpha \omega_{1_\alpha} - \omega_{3_\alpha} \wedge \omega_{2_\alpha} = (D_t^\alpha D_x^\alpha u + \sin u) dt^\alpha \wedge dx^\alpha = 0 \quad \text{iff } D_t^\alpha D_x^\alpha u + \sin u = 0$$

$$d^\alpha \omega_{2_\alpha} = -\frac{D_x^\alpha u \sin u}{\lambda} dx^\alpha \wedge dt^\alpha, \quad \omega_{1_\alpha} \wedge \omega_{3_\alpha} = -D_x^\alpha u dx^\alpha \wedge \frac{\sin u}{\lambda} dt^\alpha = -\frac{D_x^\alpha u \sin u}{\lambda} dx^\alpha \wedge dt^\alpha,$$

$$\text{then } d^\alpha \omega_{2_\alpha} - \omega_{1_\alpha} \wedge \omega_{3_\alpha} = \left(\frac{D_x^\alpha u \sin u}{\lambda} - \frac{D_x^\alpha u \sin u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0,$$

$$d^\alpha \omega_{3_\alpha} = -\frac{D_x^\alpha u \cos u}{\lambda} dx^\alpha \wedge dt^\alpha$$

$$\kappa_\alpha \omega_{1_\alpha} \wedge \omega_{2_\alpha} = \kappa_\alpha [D_x^\alpha u dx^\alpha] \wedge [\lambda dx^\alpha + \frac{\cosh u}{\lambda} dt^\alpha] = \frac{\kappa_\alpha}{\lambda} \cos u D_x^\alpha u dx^\alpha \wedge dt^\alpha$$

$$\text{then } d^\alpha \omega_{3_\alpha} = \kappa_\alpha \omega_{1_\alpha} \wedge \omega_{2_\alpha} \text{ iff } \kappa_\alpha = -1.$$

Example 3: the space-time fractional sinh–Gordon equation.

Consider

$$\omega_{1_\alpha} = D_x^\alpha u dx^\alpha, \quad \omega_{2_\alpha} = \lambda dx^\alpha + \frac{\cosh u}{\lambda} dt^\alpha, \quad \omega_{3_\alpha} = \frac{\sinh u}{\lambda} dt^\alpha, \quad (23)$$

Then M^2 is a conformable fractional pseudo-spherical surface if u satisfies the space-time fractional sinh–Gordon equation

$$D_t^\alpha D_x^\alpha u = \sinh u. \quad (24)$$

Proof:

$$d^\alpha \omega_{1_\alpha} = D_t^\alpha D_x^\alpha u dt^\alpha \wedge dx^\alpha, \quad \omega_{3_\alpha} \wedge \omega_{2_\alpha} = \frac{\sinh u}{\lambda} dt^\alpha \wedge (\lambda dx^\alpha + \frac{\cosh u}{\lambda} dt^\alpha) = \sinh u dt^\alpha \wedge dx^\alpha$$

$$\text{then } d^\alpha \omega_{1_\alpha} - \omega_{3_\alpha} \wedge \omega_{2_\alpha} = (D_t^\alpha D_x^\alpha u - \sinh u) dt^\alpha \wedge dx^\alpha = 0 \text{ iff } D_t^\alpha D_x^\alpha u - \sinh u = 0,$$

$$d^\alpha \omega_{2_\alpha} = \frac{D_x^\alpha u \sinh u}{\lambda} dx^\alpha \wedge dt^\alpha, \quad \omega_{1_\alpha} \wedge \omega_{3_\alpha} = D_x^\alpha u dx^\alpha \wedge \frac{\sinh u}{\lambda} dt^\alpha = \frac{D_x^\alpha u \sinh u}{\lambda} dx^\alpha \wedge dt^\alpha,$$

$$\text{then } d^\alpha \omega_{2_\alpha} - \omega_{1_\alpha} \wedge \omega_{3_\alpha} = \left(\frac{D_x^\alpha u \sinh u}{\lambda} - \frac{D_x^\alpha u \sinh u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0,$$

$$d^\alpha \omega_{3_\alpha} = \frac{D_x^\alpha u \cosh u}{\lambda} dx^\alpha \wedge dt^\alpha,$$

$$\kappa_\alpha \omega_{1_\alpha} \wedge \omega_{2_\alpha} = \kappa_\alpha [\lambda dx^\alpha + \frac{\cosh u}{\lambda} dt^\alpha] \wedge [D_x^\alpha u dx^\alpha] = -\frac{\kappa_\alpha}{\lambda} \cosh u D_x^\alpha u dx^\alpha \wedge dt^\alpha,$$

$$\text{then } d^\alpha \omega_{3_\alpha} - \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} = \left(\frac{D_x^\alpha u \cosh u}{\lambda} + \frac{\kappa_\alpha D_x^\alpha u \cosh u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0 \text{ iff } \kappa_\alpha = -1.$$

Example 4: the space-time fractional cos–Gordon equation

Consider

$$\omega_{1_\alpha} = D_x^\alpha u dx^\alpha, \quad \omega_{2_\alpha} = \lambda dx^\alpha + \frac{\sin u}{\lambda} dt^\alpha, \quad \omega_{3_\alpha} = \frac{\cos u}{\lambda} dt^\alpha, \quad (25)$$

Then M^2 is a conformable fractional pseudo-spherical surface if u satisfies the space-time fractional cos–Gordon equation

$$D_t^\alpha D_x^\alpha u - \cos u = 0. \quad (26)$$

Proof:

$$d^\alpha \omega_{1_\alpha} = D_t^\alpha D_x^\alpha u dt^\alpha \wedge dx^\alpha, \quad \omega_{3_\alpha} \wedge \omega_{2_\alpha} = \frac{\cos u}{\lambda} dt^\alpha \wedge (\lambda dx^\alpha + \frac{\sin u}{\lambda} dt^\alpha) = \cos u dt^\alpha \wedge dx^\alpha$$

$$\text{then } d^\alpha \omega_{1_\alpha} - \omega_{3_\alpha} \wedge \omega_{2_\alpha} = (D_t^\alpha D_x^\alpha u - \cos u) dt^\alpha \wedge dx^\alpha = 0 \text{ iff } D_t^\alpha D_x^\alpha u - \cos u = 0,$$

$$d^\alpha \omega_{2_\alpha} = \frac{D_x^\alpha u \cos u}{\lambda} dx^\alpha \wedge dt^\alpha, \quad \omega_{1_\alpha} \wedge \omega_{3_\alpha} = D_x^\alpha u dx^\alpha \wedge \frac{\cos u}{\lambda} dt^\alpha = \frac{D_x^\alpha u \cos u}{\lambda} dx^\alpha \wedge dt^\alpha,$$

$$\text{then } d^\alpha \omega_{2_\alpha} - \omega_{1_\alpha} \wedge \omega_{3_\alpha} = \left(\frac{D_x^\alpha u \cos u}{\lambda} - \frac{D_x^\alpha u \cos u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0,$$

$$d^\alpha \omega_{3_\alpha} = -\frac{D_x^\alpha u \sin u}{\lambda} dx^\alpha \wedge dt^\alpha$$

$$-\kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} = -\kappa_\alpha \left[\lambda dx^\alpha + \frac{\sin u}{\lambda} dt^\alpha \right] \wedge [D_x^\alpha u dx^\alpha] = -\frac{\kappa_\alpha}{\lambda} \sin u D_x^\alpha u dt^\alpha \wedge dx^\alpha = \frac{\kappa_\alpha D_x^\alpha u \sin u}{\lambda} dx^\alpha \wedge dt^\alpha$$

$$\text{then } d^\alpha \omega_{3_\alpha} - \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} = \left(\frac{D_x^\alpha u \sin u}{\lambda} + \frac{\kappa_\alpha D_x^\alpha u \sin u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0 \text{ iff } \kappa_\alpha = -1.$$

Example 5: the space-time fractional cosh–Gordon equation

Consider

$$\omega_{1_\alpha} = D_x^\alpha u dx^\alpha, \quad \omega_{2_\alpha} = \lambda dx^\alpha - \frac{\sinh u}{\lambda} dt^\alpha, \quad \omega_{3_\alpha} = -\frac{\cosh u}{\lambda} dt^\alpha, \quad (27)$$

Then M^2 is a conformable fractional pseudo-spherical surface if u satisfies the space-time fractional cosh–Gordon equation

$$D_t^\alpha D_x^\alpha u + \cosh u = 0. \quad (28)$$

Proof:

$$d^\alpha \omega_{1_\alpha} = D_t^\alpha D_x^\alpha u dt^\alpha \wedge dx^\alpha, \quad \omega_{3_\alpha} \wedge \omega_{2_\alpha} = -\frac{\cosh u}{\lambda} dt^\alpha \wedge (\lambda dx^\alpha - \frac{\sinh u}{\lambda} dt^\alpha) = -\cosh u dt^\alpha \wedge dx^\alpha$$

$$\text{then } d^\alpha \omega_{1_\alpha} - \omega_{3_\alpha} \wedge \omega_{2_\alpha} = (D_t^\alpha D_x^\alpha u + \cosh u) dt^\alpha \wedge dx^\alpha = 0, \text{ iff } D_t^\alpha D_x^\alpha u + \cosh u = 0,$$

$$\begin{aligned}
 d^\alpha \omega_{2_\alpha} &= -\frac{D_x^\alpha u \cosh u}{\lambda} dx^\alpha \wedge dt^\alpha, \quad \omega_{1_\alpha} \wedge \omega_{3_\alpha} = -D_x^\alpha u dx^\alpha \wedge \frac{\cosh u}{\lambda} dt^\alpha = -\frac{D_x^\alpha u \cosh u}{\lambda} dx^\alpha \wedge dt^\alpha, \\
 \text{then } d^\alpha \omega_{2_\alpha} - \omega_{1_\alpha} \wedge \omega_{3_\alpha} &= \left(\frac{D_x^\alpha u \cosh u}{\lambda} - \frac{D_x^\alpha u \cosh u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0, \\
 d^\alpha \omega_{3_\alpha} &= -\frac{D_x^\alpha u \sinh u}{\lambda} dx^\alpha \wedge dt^\alpha \\
 \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} &= \kappa_\alpha \left[\lambda dx^\alpha - \frac{\sinh u}{\lambda} dt^\alpha \right] \wedge [D_x^\alpha u dx^\alpha] = -\frac{\kappa_\alpha}{\lambda} \sinh u D_x^\alpha u dt^\alpha \wedge dx^\alpha = \frac{\kappa_\alpha D_x^\alpha u \sinh u}{\lambda} dx^\alpha \wedge dt^\alpha \\
 \text{then } d^\alpha \omega_{3_\alpha} - \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} &= \left(\frac{D_x^\alpha u \sinh u}{\lambda} + \frac{\kappa_\alpha D_x^\alpha u \sinh u}{\lambda} \right) dx^\alpha \wedge dt^\alpha = 0 \quad \text{iff } \kappa_\alpha = -1.
 \end{aligned}$$

Example 6: space-time fractional Nonlinear Schrödinger equation

Consider

$$\begin{aligned}
 \omega_{1_\alpha} &= \frac{1}{2}(u - \bar{u})dx^\alpha + [i\lambda(u - \bar{u}) + i(D_x^\alpha u + D_x^\alpha \bar{u})]dt^\alpha, \\
 \omega_{2_\alpha} &= \lambda dx^\alpha + [2i\lambda^2 + iu\bar{u}]dt^\alpha, \\
 \omega_{3_\alpha} &= -\frac{1}{2}(u + \bar{u})dx^\alpha + [-i\lambda(u - \bar{u}) + i(D_x^\alpha \bar{u} - D_x^\alpha u)]dt^\alpha,
 \end{aligned} \tag{29}$$

Then M^2 is a conformable fractional pseudo-spherical surface if u satisfies the space-time fractional nonlinear Schrödinger equation

$$iD_t^\alpha u + 2D_x^{\alpha\alpha} u + u^2 \bar{u} = 0. \tag{30}$$

Proof:

$$\begin{aligned}
 d^\alpha \omega_{1_\alpha} &= \frac{1}{2} d^\alpha (u - \bar{u}) \wedge dx^\alpha + d^\alpha [i\lambda(u - \bar{u}) + i(D_x^\alpha u + D_x^\alpha \bar{u})] \wedge dt^\alpha \\
 &= \frac{1}{2} (D_t^\alpha u - D_t^\alpha \bar{u}) dt^\alpha \wedge dx^\alpha + i\lambda(D_x^\alpha u - D_x^\alpha \bar{u}) + i(D_x^{\alpha\alpha} u + D_x^{\alpha\alpha} \bar{u}) dx^\alpha \wedge dt^\alpha \\
 &= \left[-\frac{1}{2} (D_t^\alpha u - D_t^\alpha \bar{u}) + i\lambda(D_x^\alpha u - D_x^\alpha \bar{u}) + i(D_x^{\alpha\alpha} u + D_x^{\alpha\alpha} \bar{u}) \right] dx^\alpha \wedge dt^\alpha, \\
 \omega_{3_\alpha} \wedge \omega_{2_\alpha} &= \left[-\frac{1}{2}(u + \bar{u})dx^\alpha + [-i\lambda(u + \bar{u}) + i(D_x^\alpha \bar{u} - D_x^\alpha u)]dt^\alpha \right] \wedge (\lambda dx^\alpha + [2i\lambda^2 + iu\bar{u}]dt^\alpha) \\
 &= -\frac{1}{2}(u + \bar{u})(2i\lambda^2 + iu\bar{u})dx^\alpha \wedge dt^\alpha + \lambda[-i\lambda(u + \bar{u}) + i(D_x^\alpha \bar{u} - D_x^\alpha u)]dt^\alpha \wedge dx^\alpha \\
 &= [-i\lambda^2(u + \bar{u}) - \frac{i}{2}u\bar{u}(u + \bar{u}) + i\lambda^2(u + \bar{u}) - i\lambda(D_x^\alpha \bar{u} - D_x^\alpha u)]dx^\alpha \wedge dt^\alpha \\
 &= \left[-\frac{i}{2}u^2 \bar{u} - \frac{i}{2}\bar{u}^2 u - i\lambda D_x^\alpha \bar{u} + i\lambda D_x^\alpha u \right] dx^\alpha \wedge dt^\alpha,
 \end{aligned}$$

$$\begin{aligned}
 d^\alpha \omega_{1_\alpha} - \omega_{3_\alpha} \wedge \omega_{2_\alpha} &= \left[-\frac{1}{2} (D_t^\alpha u - D_t^\alpha \bar{u}) + i(D_x^{\alpha\alpha} u + D_x^{\alpha\alpha} \bar{u}) + \frac{i}{2} u^2 \bar{u} + \frac{i}{2} \bar{u}^2 u \right] dx^\alpha \wedge dt^\alpha \\
 &= \left[\frac{i}{2} (iD_t^\alpha u + 2D_x^{\alpha\alpha} u + u^2 \bar{u}) + \frac{i}{2} (iD_t^\alpha \bar{u} + 2D_x^{\alpha\alpha} \bar{u} + \bar{u}^2 u) \right] dx^\alpha \wedge dt^\alpha = 0,
 \end{aligned}$$

$$\text{if } iD_t^\alpha u + 2D_x^{\alpha\alpha} u + u^2 \bar{u} = 0, \quad iD_t^\alpha \bar{u} + 2D_x^{\alpha\alpha} \bar{u} + \bar{u}^2 u = 0$$

$$d^\alpha \omega_{2_\alpha} = id^\alpha(u\bar{u}) \wedge dt^\alpha = i(uD_x^\alpha \bar{u} + \bar{u}D_x^\alpha u) dx^\alpha \wedge dt^\alpha$$

$$\begin{aligned}
 \omega_{1_\alpha} \wedge \omega_{3_\alpha} &= \left[\frac{1}{2} (u - \bar{u}) dx^\alpha + [i\lambda(u - \bar{u}) + i(D_x^\alpha u + D_x^\alpha \bar{u})] dt^\alpha \right] \wedge \\
 &\quad \left[-\frac{1}{2} (u + \bar{u}) dx^\alpha + [-i\lambda(u + \bar{u}) + i(D_x^\alpha \bar{u} - D_x^\alpha u)] dt^\alpha \right] \\
 &= \left[\frac{1}{2} (u - \bar{u}) [-i\lambda(u + \bar{u}) + i(D_x^\alpha \bar{u} - D_x^\alpha u)] + [i\lambda(u - \bar{u}) + i(D_x^\alpha u + D_x^\alpha \bar{u})] \left[\frac{1}{2} (u + \bar{u}) \right] \right] dx^\alpha \wedge dt^\alpha \\
 &= \left[-\frac{i\lambda}{2} (u - \bar{u})(u + \bar{u}) + \frac{i}{2} (u - \bar{u})(D_x^\alpha \bar{u} - D_x^\alpha u) + \frac{i\lambda}{2} (u - \bar{u})(u + \bar{u}) + \frac{i}{2} (u + \bar{u})(D_x^\alpha \bar{u} + D_x^\alpha u) \right] dx^\alpha \wedge dt^\alpha \\
 &= \left[\frac{i}{2} (u - \bar{u})(D_x^\alpha \bar{u} - D_x^\alpha u) + \frac{i}{2} (u + \bar{u})(D_x^\alpha \bar{u} + D_x^\alpha u) \right] dx^\alpha \wedge dt^\alpha = i(uD_x^\alpha \bar{u} + \bar{u}D_x^\alpha u) dx^\alpha \wedge dt^\alpha
 \end{aligned}$$

$$d^\alpha \omega_{2_\alpha} - \omega_{1_\alpha} \wedge \omega_{3_\alpha} = [i(uD_x^\alpha \bar{u} + \bar{u}D_x^\alpha u) - i(uD_x^\alpha \bar{u} + \bar{u}D_x^\alpha u)] dx^\alpha \wedge dt^\alpha = 0,$$

$$d^\alpha \omega_{3_\alpha} - \kappa_\alpha \omega_{2_\alpha} \wedge \omega_{1_\alpha} = \left[\frac{i}{2} (iD_t^\alpha u + 2D_x^{\alpha\alpha} u - \kappa_\alpha u^2 \bar{u}) + \frac{i}{2} (iD_t^\alpha \bar{u} + 2D_x^{\alpha\alpha} \bar{u} - \kappa_\alpha \bar{u}^2 u) \right] dx^\alpha \wedge dt^\alpha = 0,$$

$$\text{if } \kappa_\alpha = -1 \text{ and } iD_t^\alpha u + 2D_x^{\alpha\alpha} u + u^2 \bar{u} = 0, \quad iD_t^\alpha \bar{u} + 2D_x^{\alpha\alpha} \bar{u} + \bar{u}^2 u = 0.$$

When $\alpha = 1$, equation (29) is similar to equation (13) in (Khater et al., 1997).

4. QUASI CONSERVATION LAWS FOR A FRACTIONAL NONLINEAR EVOLUTION EQUATION

Consider the fractional differential equation

$$D_t^\alpha u = F(u, D_x^\alpha u, D_x^{\alpha\alpha} u, D_x^{\alpha\alpha\alpha} u, \dots), \quad (31)$$

realized by the fractional quasi AKNS-like system and conformable fractional pseudo-spherical surfaces as

$$\begin{aligned}
 D_x^\alpha \Psi &= U\Psi, \quad D_t^\alpha \Psi = V\Psi, \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \\
 U &= \begin{pmatrix} \frac{\eta}{2} & q(x, t) \\ r(x, t) & -\frac{\eta}{2} \end{pmatrix}, \quad V = \begin{pmatrix} A(x, t) & B(x, t) \\ C(x, t) & -A(x, t) \end{pmatrix}, \quad (32)
 \end{aligned}$$

or in a simple form as

$$\begin{aligned} D_x^\alpha A &= q C - r B, \\ D_x^\alpha B - \eta B &= D_t^\alpha q - 2 A q, \\ D_x^\alpha C + \eta C &= D_t^\alpha r + 2 A r. \end{aligned} \quad (33)$$

and the fractional matrix one-form Ω_α is

$$\Omega_\alpha = \frac{1}{2} \begin{pmatrix} \omega_{2_\alpha} & \omega_{1_\alpha} - \omega_{3_\alpha} \\ \omega_{1_\alpha} + \omega_{3_\alpha} & -\omega_{2_\alpha} \end{pmatrix} = U d^\alpha x + V d^\alpha t. \quad (34)$$

By setting $\varphi_1 = \frac{q}{\Gamma}$, $\varphi_2 = r \Gamma$, and consider $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is a nontrivial solution of the fractional

linear system $d^\alpha v = \Omega_\alpha v$, then φ_1 and φ_2 satisfies the following fractional differential systems

$$D_x^\alpha \varphi_1 = q r + \left(\frac{D_x^\alpha q}{q} - \eta \right) \varphi_1 - \varphi_1^2, \quad D_t^\alpha \left(\frac{\eta}{2} + \varphi_1 \right) = D_x^\alpha \left(A + \frac{B}{q} - \varphi_1 \right), \quad (35)$$

and

$$D_x^\alpha \varphi_2 = -q r + \left(\frac{D_x^\alpha r}{r} + \eta \right) \varphi_2 + \varphi_2^2, \quad D_t^\alpha \left(\frac{\eta}{2} + \varphi_2 \right) = D_x^\alpha \left(A + \frac{C}{q} + \varphi_2 \right), \quad (36)$$

This fractional system provides one with at least one η -dependent fractional quasi conservation law of the fractional differential equation (32). From equation (35) or equation (36), one obtains a sequence of η -independent conservation laws by expanding φ_1 or φ_2 in inverse powers of η

$$\varphi_2 = \sum_{n=1}^{\infty} \varphi_2^{(n)} \eta^{-n}, \quad (37)$$

We have the recursion relation

$$\varphi_2^{(1)} = -q r, \quad \varphi_2^{(n+1)} = \frac{D_x^\alpha r}{r} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{i=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1. \quad (38)$$

The procedure is clarified in the following examples.

Example 1: space-time fractional Liouville's equation.

Consider

$$\begin{aligned} \omega_{1_\alpha} &= \frac{d^\alpha u}{dx^\alpha} dx^\alpha, & \omega_{2_\alpha} &= \lambda dx^\alpha + \frac{e^u}{\lambda} dt^\alpha, & \omega_{3_\alpha} &= \frac{e^u}{\lambda} dt^\alpha, \\ q = r &= \frac{1}{2} D_x^\alpha u, & A = C &= \frac{e^u}{2\lambda}, & B &= -\frac{e^u}{2\lambda}, \end{aligned} \quad (39)$$

Equation (36) becomes

$$D_x^\alpha \varphi_2 = -\frac{1}{4} \left(D_x^\alpha u \right)^2 + \left(\frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} + \eta \right) \varphi_2 + \varphi_2^2, \quad (40)$$

Assume that φ_2 can be expanded in a series of the form (37). Equation (40) implies that φ_2 is determined by the recursion relation

$$\varphi_2^{(1)} = -\frac{1}{4} \left(D_x^\alpha u \right)^2, \quad (41)$$

$$\varphi_2^{(n+1)} = \frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{\eta=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1, \quad (42)$$

and the coefficients of the series in η , given in (42) are a sequence of conserved densities for the space-time fractional Liouville's equation.

Example 2: the space-time fractional sine–Gordon equation.

Consider

$$\begin{aligned} \omega_{1_\alpha} &= D_x^\alpha u dx^\alpha, & \omega_{2_\alpha} &= \lambda dx^\alpha + \frac{\cos u}{\lambda} dt^\alpha, & \omega_{3_\alpha} &= -\frac{\sin u}{\lambda} dt^\alpha, \\ q=r &= \frac{1}{2} D_x^\alpha u, & A &= \frac{\cos u}{2\lambda}, & B &= \frac{\sin u}{2\lambda}, & C &= -\frac{\sin u}{2\lambda}, \end{aligned} \quad (43)$$

Equation (36) becomes

$$D_x^\alpha \varphi_2 = -\frac{1}{4} \left(D_x^\alpha u \right)^2 + \left(\frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} + \eta \right) \varphi_2 + \varphi_2^2, \quad (44)$$

Assume that φ_2 can be expanded in a series of the form (37). Equation (44) implies that φ_2 is determined by the recursion relation

$$\varphi_2^{(1)} = -\frac{1}{4} \left(D_x^\alpha u \right)^2, \quad (45)$$

$$\varphi_2^{(n+1)} = \frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{\eta=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1, \quad (46)$$

and the coefficients of the series in η , given in (46) are a sequence of conserved densities for the space-time fractional sine–Gordon equation.

Example 3: the space-time fractional sinh–Gordon equation.

Consider

$$\begin{aligned} \omega_{1_\alpha} &= D_x^\alpha u dx^\alpha, & \omega_{2_\alpha} &= \lambda dx^\alpha + \frac{\cosh u}{\lambda} dt^\alpha, & \omega_{3_\alpha} &= \frac{\sinh u}{\lambda} dt^\alpha, \\ q=r &= \frac{1}{2} D_x^\alpha u, & A &= \frac{\cosh u}{2\lambda}, & B &= -\frac{\sinh u}{2\lambda}, & C &= \frac{\sinh u}{2\lambda} \end{aligned} \quad (47)$$

Equation (36) becomes

$$D_x^\alpha \varphi_2 = -\frac{1}{4} \left(D_x^\alpha u \right)^2 + \left(\frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} + \eta \right) \varphi_2 + \varphi_2^2, \quad (48)$$

Assume that φ_2 can be expanded in a series of the form (37). Equation (48) implies that φ_2 is determined by the recursion relation

$$\varphi_2^{(1)} = -\frac{1}{4} \left(D_x^\alpha u \right)^2, \quad (49)$$

$$\varphi_2^{(n+1)} = \frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{i=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1, \quad (50)$$

and the coefficients of the series in η , given in (50) are a sequence of conserved densities.

Example 4: the space-time fractional cos–Gordon equation.

Consider

$$\begin{aligned} \omega_{1_\alpha} &= D_x^\alpha u dx^\alpha, & \omega_{2_\alpha} &= \lambda dx^\alpha + \frac{\sin u}{\lambda} dt^\alpha, & \omega_{3_\alpha} &= \frac{\cos u}{\lambda} dt^\alpha, \\ q = r &= \frac{1}{2} D_x^\alpha u, & A &= \frac{\sin u}{2\lambda}, & B &= -\frac{\cos u}{2\lambda}, & C &= \frac{\cos u}{2\lambda} \end{aligned} \quad (51)$$

Equation (36) becomes

$$D_x^\alpha \varphi_2 = -\frac{1}{4} \left(D_x^\alpha u \right)^2 + \left(\frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} + \eta \right) \varphi_2 + \varphi_2^2, \quad (52)$$

Assume that φ_2 can be expanded in a series of the form (37). Equation (52) implies that φ_2 is determined by the recursion relation

$$\varphi_2^{(1)} = -\frac{1}{4} \left(D_x^\alpha u \right)^2, \quad (53)$$

$$\varphi_2^{(n+1)} = \frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{i=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1, \quad (54)$$

and the coefficients of the series in η , given in (54) are a sequence of conserved densities.

Example 5: the space-time fractional cosh–Gordon equation.

Consider

$$\begin{aligned} \omega_{1_\alpha} &= D_x^\alpha u dx^\alpha, & \omega_{2_\alpha} &= \lambda dx^\alpha - \frac{\sinh u}{\lambda} dt^\alpha, & \omega_{3_\alpha} &= -\frac{\cosh u}{\lambda} dt^\alpha, \\ q = r &= \frac{1}{2} D_x^\alpha u, & A &= -\frac{\sinh u}{2\lambda}, & B &= \frac{\cosh u}{2\lambda}, & C &= -\frac{\cosh u}{2\lambda} \end{aligned} \quad (55)$$

Equation (36) becomes

$$D_x^\alpha \varphi_2 = -\frac{1}{4} \left(D_x^\alpha u \right)^2 + \left(\frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} + \eta \right) \varphi_2 + \varphi_2^2, \quad (56)$$

Assume that φ_2 can be expanded in a series of the form (37). Equation (56) implies that φ_2 is determined by the recursion relation

$$\varphi_2^{(1)} = -\frac{1}{4} \left(D_x^\alpha u \right)^2, \quad (57)$$

$$\varphi_2^{(n+1)} = \frac{D_x^{\alpha\alpha} u}{D_x^\alpha u} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{i=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1, \quad (58)$$

and the coefficients of the series in η , given in (58) are a sequence of conserved densities.

Example 6: the space-time fractional Nonlinear Schrödinger equation.

Consider

$$\begin{aligned} \omega_{1_\alpha} &= \frac{1}{2} (u - \bar{u}) dx^\alpha + [i \lambda (u - \bar{u}) + i (D_x^\alpha u + D_x^\alpha \bar{u})] dt^\alpha, \\ \omega_{2_\alpha} &= \lambda dx^\alpha + [2i \lambda^2 + i u \bar{u}] dt^\alpha, \\ \omega_{3_\alpha} &= -\frac{1}{2} (u + \bar{u}) dx^\alpha + [-i \lambda (u - \bar{u}) + i (D_x^\alpha \bar{u} - D_x^\alpha u)] dt^\alpha, \\ q &= \frac{1}{2} u, \quad r = -\frac{1}{2} \bar{u}, \quad A = -i \eta^2 + \frac{1}{2} i u \bar{u}, \quad B = i \eta u + i D_x^\alpha u, \quad C = -i \eta \bar{u} + i D_x^\alpha \bar{u}, \end{aligned} \quad (59)$$

Equation (36) becomes

$$D_x^\alpha \varphi_2 = -\frac{1}{4} u \bar{u} + \left(\frac{D_x^\alpha \bar{u}}{\bar{u}} + \eta \right) \varphi_2 + \varphi_2^2, \quad (60)$$

Assume that φ_2 can be expanded in a series of the form (37). Equation (60) implies that φ_2 is determined by the recursion relation

$$\varphi_2^{(1)} = -\frac{1}{4} u \bar{u}, \quad (61)$$

$$\varphi_2^{(n+1)} = \frac{D_x^\alpha \bar{u}}{\bar{u}} \varphi_2^{(n)} + D_x^\alpha \varphi_2^{(n)} + \sum_{i=1}^{\infty} \varphi_2^{(i)} \varphi_2^{(n-i)}, \quad n \geq 1, \quad (62)$$

and the coefficients of the series in η , given in (62) are a sequence of conserved densities.

5: CONCLUSION AND DISCUSSION

This paper is concerned with the construction of the geometrical interpretation of the conformable fractional pseudo-spherical surfaces. Also, the relationship between the fractional quasi AKNS-like system and conformable fractional pseudo-spherical surfaces are discussed. Besides, we constructed the fractional quasi conservation laws for the considered fractional equations from the defined

fractional quasi AKNS-like system and the fractional matrix one-form. The fractional Levi-Civita connection one-form on the fTM^2 with respect to the fractional moving frame and the fractional Gauss equation are discussed. Also, some examples of different fractional matrix one-forms are formulated. Besides, the moving frame and the associated linear scattering problems are constructed. We analysed the space-time fractional versions of the Liouville equation, the sine-Gordon equation, the sinh-Gordon equation, the cos-Gordon equation, the cosh-Gordon equation, and the nonlinear Schrödinger equation. Equation (16) represents the fractional quasi zero curvature conditions which has a Lie bracket. It leads us to Poisson bracket which is infinite-dimensional in quantum mechanics when $\alpha = 1$ has a natural quantum analog describing a quantum system of an infinite number of particles interaction pairwise via their potentials. This means that equation (16) is the key to generalize the fractional zero curvature condition in quantum mechanics.

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