

Bayesian Estimation of Weibull Rayleigh Distribution Under Balanced Loss Functions.

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ABSTRACT

In this paper, we have derived the expressions for bayesian estimate of unknown parameter of the Weibull Rayleigh distribution. The results of the bayes estimates are utilized to obtained the estimates under different loss functions. MCMC techniques with in-built Gibbs sampling are used to obtain samples from the unknown parameter. 95% confidence intervals have also been derived for precisional purposes. Finally, a real life data has been used for simulation purposes. Monte Carlo Simulation has also been performed to find efficient estimation method and better estimates among the Loss function.

Keywords: Weibull Rayleigh Distribution, Markov Chain Monte Carlo Technique, Bayesian Estimation, Balanced Loss Function.

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1 Introduction

Various probability distributions have been utilized for modelling in different areas for applied mathematics, natural sciences and engineering. For these reasons, several methods for generating new distributions has been introduced by different researchers and practitioners. As already known, the weibull distribution and Rayleigh distribution are two important distributions in statistical and mathematical analysis. Hence, a new generalization of Weibull and Rayleigh distribution called Weibull- Rayleigh distribution has been presented by (Marovci and Elbatal, 2015). In this paper, we have derived the bayesian estimates of the unknown parameter of Weibull Rayleigh distribution by using Markov Chain Monte Carlo Method. These method helps us to obtain samples from unknown parameter efficiently. It only requires the conditional posterior distribution for each unknown parameter in order to obtain samples. Here, we applied the Metropolis Hastings algorithm to obtain samples of the unknown parameters from their conditional distributions generated by bayesian inference from the Weibull Rayleigh dsitribution. (Marcelo, Silva and Cordeiro, 2014) derived the general mathematical properties for newly

generated Weibull family of distribution. This family was further used to extend various well known distributions such as Uniform distribution, Rayleigh distribution, Gamma distribution, Exponential distribution etc. As a result, from this generator (Marovci and Elbatal, 2015) has generated a new generalization of Weibull and Rayleigh distribution called Weibull-Rayleigh Distribution (WR Distribution). They also studied the general mathematical properties of WR distribution. A lot of work has been done by many researchers on estimating the bayesian estimates by MCMC for example (EL-Sagheer, 2018) estimated the parameters of Weibull Gamma distribution based on progressively censored samples, (Mahmoud and Ghazal, 2018) estimated the exponentiated rayleigh distribution based on generalized type II hybrid censored data etc. (Fatima, 2017) estimated the parameters of WR distribution using bayesian analysis under different loss functions. (Kumar, Kour and Kour, 2018) estimated the probability density function of Lomax Distribution in which MLE and UMVUE of the shape parameter and probability density function have been estimated. (Hemeda and Mohamed, 2018) developed weibull exponentiated inverted rayleigh distribution with its properties and applications. (Gupta and Gupta, 2018) studied sequential analysis of unknown parameter of exponentiated inverted weibull distribution. This paper comprises of estimating the bayesian estimates of parameters of weibull rayleigh distribution under different loss functions by using Metropolis Hastings algorithm. The Cumulative distribution function of the Weibull Rayleigh distribution is given by

$$F(x) = 1 - e^{-\alpha(e^{\frac{\theta}{2}x^2} - 1)^\beta}; x > 0 \text{ and } \alpha, \beta, \theta > 0 \quad (1.1)$$

and the corresponding probability density function of weibull rayleigh distribution is given by

$$f(x) = \alpha\beta\theta x e^{\frac{\theta}{2}x^2} (e^{\frac{\theta}{2}x^2} - 1)^{\beta-1} e^{-\alpha(e^{\frac{\theta}{2}x^2} - 1)^\beta}; x > 0 \text{ and } \alpha, \beta, \theta > 0 \quad (1.2)$$

The Reliability Function of the Weibull Rayleigh Distribution is

$$R(t) = 1 - F(t) = e^{-\alpha(e^{\frac{\theta}{2}t^2} - 1)^\beta}; x > 0 \text{ and } \alpha, \beta, \theta > 0 \quad (1.3)$$

2 Maximum Likelihood Estimation for Unknown Parameters α, β and θ Weibull Rayleigh distribution

On the lines of (Marovci and Elbatal, 2015), consider a random sample $X_1, X_2, \dots, X_n$ from the weibull rayleigh distribution. The log likelihood of the distribution is obtained as

$$\begin{aligned} \log L(x, \alpha, \beta, \theta) &= n \log(\alpha) + n \log(\beta) + n \log(\theta) + \sum_{i=1}^n \log x_i + \frac{\theta}{2} \sum_{i=1}^n x_i^2 \\ &\quad + (\beta - 1) \sum_{i=1}^n \log(e^{\frac{\theta}{2}x_i^2} - 1) - \alpha \sum_{i=1}^n \log[e^{\frac{\theta}{2}x_i^2} - 1]^\beta \end{aligned}$$

Differentiating the above equation with respect to each parameter α, β and θ and equating to zero, we obtain

$$\hat{\alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \log[e^{\frac{\theta}{2}x_i^2} - 1]^\beta = 0 \quad (2.1)$$

$$\hat{\beta} = \frac{n}{\beta} + \sum_{i=1}^n \log [e^{\frac{\theta}{2} x_i^2} - 1]^\beta - \alpha \sum_{i=1}^n [e^{\frac{\theta}{2} x_i^2} - 1]^\beta \log \sum_{i=1}^n \log [e^{\frac{\theta}{2} x_i^2} - 1]^\beta = 0 \quad (2.2)$$

$$\hat{\theta} = \frac{n}{\theta} + \frac{1}{2} \sum_{i=1}^n x_i^2 + (\beta - 1) \sum_{i=1}^n \frac{x_i^2 e^{\frac{\theta}{2} x_i^2}}{(e^{\frac{\theta}{2} x_i^2} - 1)} - \frac{\alpha \beta}{2} \sum_{i=1}^n x_i^2 e^{\frac{\theta}{2} x_i^2} (e^{\frac{\theta}{2} x_i^2} - 1)^{\beta-1} = 0 \quad (2.3)$$

As we observed, the above equations of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\theta}$ have no closed form, so we have to solve these equations numerically by using any numerical method for example Newton Raphson method.

3 Bayesian Estimation of Unknown Parameter using MCMC Approach

In this section, bayesian estimates of α , β and θ by using Metropolis Hasting techniques and their corresponding credible intervals have been obtained. We assumed that the parameters are independent of each other and follows gamma prior distributions which is given by

$$\begin{aligned} \pi_1(\alpha|a_1, b_1) &\propto \alpha^{a_1-1} e^{-b_1 \alpha}; \alpha > 0 \\ \pi_2(\beta|a_2, b_2) &\propto \beta^{a_2-1} e^{-b_2 \beta}; \beta > 0 \\ \pi_3(\theta|a_3, b_3) &\propto \theta^{a_3-1} e^{-b_3 \theta}; \theta > 0 \end{aligned}$$

where a_i and $b_i, i = 1, 2, 3$ are non negative hyperparameters and known. The Joint prior distribution for α, β and θ is given by

$$\pi(\alpha, \beta, \theta) \propto \alpha^{a_1-1} \beta^{a_2-1} \theta^{a_3-1} e^{-b_1 \alpha} e^{-b_2 \beta} e^{-b_3 \theta}; \alpha, \beta, \theta > 0$$

The joint posterior density function of parameters α, β and θ is obtained as

$$h^*(\alpha, \beta, \theta|\underline{x}) = \frac{L(\alpha, \beta, \theta|\underline{x}) \times \pi_1(\alpha) \times \pi_1(\beta) \times \pi_1(\theta)}{\int_0^\infty \int_0^\infty \int_0^\infty L(\alpha, \beta, \theta|\underline{x}) \times \pi_1(\alpha) \times \pi_1(\beta) \times \pi_1(\theta)} \quad (3.1)$$

the above equation cannot be obtained in closed form so here in order to find the bayes estimates of the parameters we have used MCMC to generate samples from the joint posterior density function. The joint posterior density function can be written as

$$h^*(\alpha, \beta, \theta|\underline{x}) \propto \alpha^{n+a_1-1} \beta^{n+a_2-1} \theta^{n+a_3-1} e^{-b_1 \alpha} e^{-b_2 \beta} e^{-\theta(b_3 - \frac{1}{2} \sum_{i=1}^n x_i^2)} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2} x_i^2} - 1)} e^{-\alpha[b_1 + \sum_{i=1}^n (e^{\frac{\theta}{2} x_i^2} - 1)^\beta]} \quad (3.2)$$

The conditional posterior densities of α, β and θ can be written as

$$\begin{aligned} h_\alpha(\alpha|\beta, \theta, \underline{x}) &\propto \alpha^{n+a_1-1} e^{-\alpha[b_1 + \sum_{i=1}^n (e^{\frac{\theta}{2} x_i^2} - 1)^\beta]} \\ h_\beta(\beta|\alpha, \theta, \underline{x}) &\propto \beta^{n+a_2-1} e^{-b_2 \beta} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2} x_i^2} - 1)} e^{-\alpha[\sum_{i=1}^n (e^{\frac{\theta}{2} x_i^2} - 1)^\beta]} \\ h_\theta(\theta|\alpha, \beta, \underline{x}) &\propto \theta^{n+a_3-1} e^{-\theta(b_3 - \frac{1}{2} \sum_{i=1}^n x_i^2)} e^{(\beta-1) \sum_{i=1}^n \log(e^{\frac{\theta}{2} x_i^2} - 1)} e^{-\alpha[\sum_{i=1}^n (e^{\frac{\theta}{2} x_i^2} - 1)^\beta]} \end{aligned}$$

It is observed that the conditional density of α follows gamma distribution with shape parameter $n + a_1$ and scale parameter $[b_1 + \sum_{i=1}^n (e^{\frac{\theta}{2} x_i^2} - 1)^\beta]$. Hence the samples for α are obtained by using gamma distribution. The conditional densities of β and θ donot follow any particular distribution, so it is quite difficult to draw samples from them directly. In this case, we generate samples for β and θ by Metropolis Hasting Algorithm by using normal proposal distributions.

Here we generate N samples from α, β and θ . In order to remove the effect cause by the selection of initial values of α, β and θ , the first K simulated values from the generated samples should have been discarded. The remaining samples values from $\alpha^{(t)}, \beta^{(t)}$ and $\theta^{(t)}, t = K + 1, \dots, N$, for sufficiently large N , forms an approximate posterior samples which can be used to develop the bayes estimates of α, β and θ given as

$$\begin{aligned}\hat{\alpha}_{MC} &= \frac{1}{N - K} \sum_{t=K+1}^N \alpha^{(t)} \\ \hat{\beta}_{MC} &= \frac{1}{N - K} \sum_{t=K+1}^N \beta^{(t)} \\ \hat{\theta}_{MC} &= \frac{1}{N - K} \sum_{t=K+1}^N \theta^{(t)}\end{aligned}$$

where N is the total sample generated from the conditional posterior distribution and K is the burn-in samples. To compute the credible intervals of the parameters α, β and θ , sort the samples of the parameters generated by MH algorithm in increasing order and then compute the $100(1 - \gamma)\%$ credibles intervals of α, β and θ as

$$\begin{aligned}&[\alpha^{(N-K)(\frac{\gamma}{2})}, \alpha^{(N-K)(1-\frac{\gamma}{2})}] \\ &[\beta^{(N-K)(\frac{\gamma}{2})}, \beta^{(N-K)(1-\frac{\gamma}{2})}] \\ &[\theta^{(N-K)(\frac{\gamma}{2})}, \theta^{(N-K)(1-\frac{\gamma}{2})}]\end{aligned}$$

4 Bayesian Estimation of Unknown Parameter using MCMC Approach Under Different Loss Function

An asymmetric Loss Functions plays an important role in Bayesian inference and statistical analysis by making it praticable and applicable in different areas. The various Loss Fucntions we have used in this analysis are BSELF and Balanced LINEX Loss Function. The LINEX loss function is one of the popular loss function among various loss functions and was introduced by (Varian, 1975), and several others; among of them (Basu and Ebrahimi, 1991). (Jozani, Marchand and Parsian, 2012) developed a more generalized form of Loss Function called Balanced Loss Function of the form

$$L_{\rho, \omega, \delta_0}(\theta, \delta) = \omega \rho(\delta, \delta_0) + (1 - \omega) \rho(\theta, \delta)$$

Where $\rho(\theta, \delta)$ is an arbitrary loss function, while δ_0 is the choosen priori target estimator of θ , obtained for instance using the criterion of maximum likelihood, least-squares or unbiasedness.

Loss $L_{\rho, \omega, \delta_0}$, which depends on the observed value of $\delta_0(X)$, reflects a desire of closeness of δ to both: (i) the target estimator δ_0 , and (ii) the unknown parameter θ ; with the relative importance of these criteria governed by the choice of $\omega[0, 1)$.

A commonly used loss function in bayesian statistics is Square Error Loss Function(SELF) of the form

$$L(\theta, \delta) = c(\theta - \delta)^2$$

which is a symmetric loss function. (Ahmad, Jafari, E. and A., 2009) generate a Balanced Square Error Loss Function by using arbitrary loss function $\rho(\theta, \delta) = (\theta - \delta)^2$ of the form

$$L_{\omega, \delta_0}(\theta, \delta) = \omega(\delta - \delta_0)^2 + (1 - \omega)(\delta - \theta)^2$$

Hence the approximate bayesian estimate under the balanced square error loss function after performing K iterations of an MCMC algorithm is obtained as

$$\hat{\theta}_{BS} = \omega\hat{\theta} + \frac{1 - \omega}{N - K} \sum_{t=K+1}^N \theta^{(t)}$$

Also by using the LINEX loss function $\rho(\theta, \delta) = e^{a(\delta - \theta)} - a(\delta - \theta) - 1; a \neq 0$, we get the approximate bayes estimate of the BLINEX loss function as

$$\hat{\theta}_{BL} = -\frac{1}{a} \log \left(\omega e^{-a\hat{\theta}} + \frac{1 - \omega}{N - K} \sum_{t=K+1}^N e^{-a\theta^{(t)}} \right)$$

5 Application to Real Life Data

This section represents the estimation of real life data given by (Wingo, 1983) which is used for illustrative purposes. The data represents the relief hours for 50 arthritic patients. This data was recently used by (Soliman, Abd Ellah, Essam and Farghal, 2015) for the bayesian estimation of Exponentiated Frechet Model by using MCMC approach. The data set for the relief hours of 50 arthritic patients is given in the below table. Now by using the Metropolis Hasting Algorithm, we

Table 1: Relief hours for 50 arthritic patients

0.70	0.84	0.58	0.50	0.55	0.82	0.59	0.71	0.72	0.61
0.62	0.49	0.54	0.36	0.36	0.71	0.35	0.64	0.84	0.55
0.59	0.29	0.75	0.46	0.46	0.60	0.60	0.36	0.52	0.68
0.80	0.55	0.84	0.34	0.34	0.70	0.49	0.56	0.71	0.61
0.57	0.73	0.75	0.44	0.44	0.81	0.80	0.87	0.29	0.50

can draw samples from the unknown parameters of the distribution and then use these samples for further purposes. The Metropolis Hastings Algorithm use to generate samples is given below:

1. Take the initials for $\alpha, \beta, \theta, K = \text{Burn-in}$.

2. Set $t=1$.
3. Since the posterior distribution of α follows Gamma distribution so generate $\alpha^{(t)}$ using $Gamma(n + a_1, b_1 + \sum_{i=1}^n (e^{\frac{\theta}{2} x_i^2} - 1)^\beta)$.
4. Now the posterior distribution of β and θ donot follows any particular distribution so according to Metropolis Hastings, we can generate $\beta^{(t)}$ and $\theta^{(t)}$ from $h_\beta(\beta|\alpha, \theta, \underline{x})$ and $h_\theta(\theta|\alpha, \beta, \underline{x})$ by using $N(\beta^{(t-1)}, \sigma)$ and $N(\theta^{(t-1)}, \sigma)$ as the proposal distribution.
5. Put $t = t + 1$.
6. Repeat the steps from 3-5 N times and get $\alpha^{(t)}, \beta^{(t)}$ and $\theta^{(t)}, t = K + 1 \dots N$.

To ensure convergence and to remove the affect of the selection of initial values, the first K simulated variates are discarded. Then the samples selected for inference are $\alpha^{(t)}, \beta^{(t)}$ and $\theta^{(t)}, t = K + 1 \dots N$, for sufficiently large N, forms approximate posterior samples which can be used for bayesian inference. By using these posterior samples, we can obtain Bayes estimates of the unknown parameters α, β, θ .

Table 2: MLE and Bayesian Estimates of unknown paramters under different loss function

parameters	MLE	ω	BSEL	BLINEX	
				$q = -1$	$q = 1$
α	4.3169	0.0	4.7420	8.3887	4.0053
		0.3	4.6067	7.0597	4.0901
		0.6	4.4940	7.1263	4.1920
β	1.8051	0.0	1.4533	1.4585	1.4478
		0.3	1.5596	1.5762	1.6801
		0.6	1.6638	1.5433	1.6457
θ	1.7066	0.0	1.4637	1.5330	1.3360
		0.3	1.5363	1.5947	1.4221
		0.6	1.6101	1.6414	1.5424

Table 3: 95% Confidence Interval and Credible Interval For α, β, θ

parameters	Confidence Interval		Credible interval	
	Lower Limit	Upper Limit	Lower Limit	Upper Limit
α	-23.7101	32.3438	0.6270	8.8912
β	0.9879	2.6224	1.1853	1.7178
θ	-2.8562	6.2695	0.4285	2.5018

By using the convention that this data is from Weibull Rayleigh Distribution, we generate a markov chain of 10,000 observations from the posterior marginal densities of α, β and θ . The first 2000 observations are burned in order to check convergence in the samples. Trace plots are plotted to check the convergence graphically. Here we use infromative priors for α, β and θ i.e., when hyperparamters are known $a'_i s = 2$ and $b'_i s = 3, i = 1, 2, 3$.

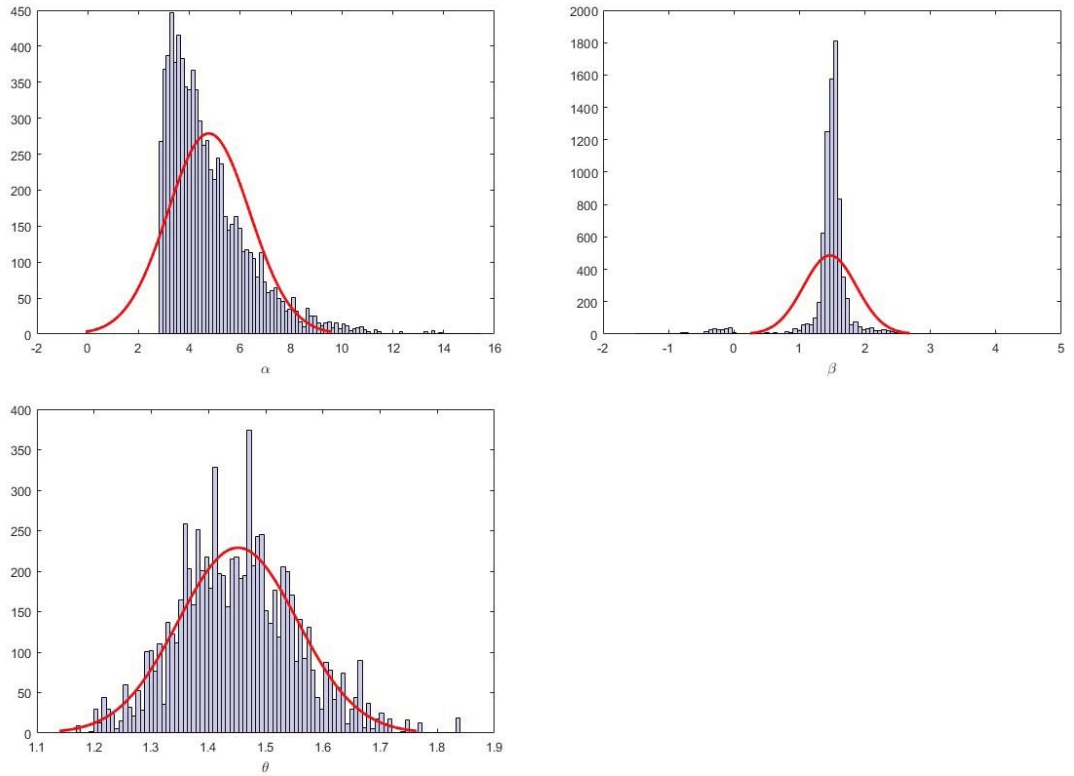


Figure 1: Histogram for α, β and θ of Weibull Rayleigh Distribution

Table 2 provides the Maximum Likelihood estimates and Bayesian Estimates under BSEL and BLINEX loss function for parameters of the distribution. Table 3 gives us the 95% probability intervals for different parameters. Figure 1 represents the histogram of the samples of unknown parameters α, β and θ generated from Gibbs sampler within Metropolis Hastings sampler. From the histogram, we can see that samples generated from conditional posterior distribution are almost symmetrical. Figure 2 represents the Markov Chain Path of α, β and θ for the 10,000 observations.

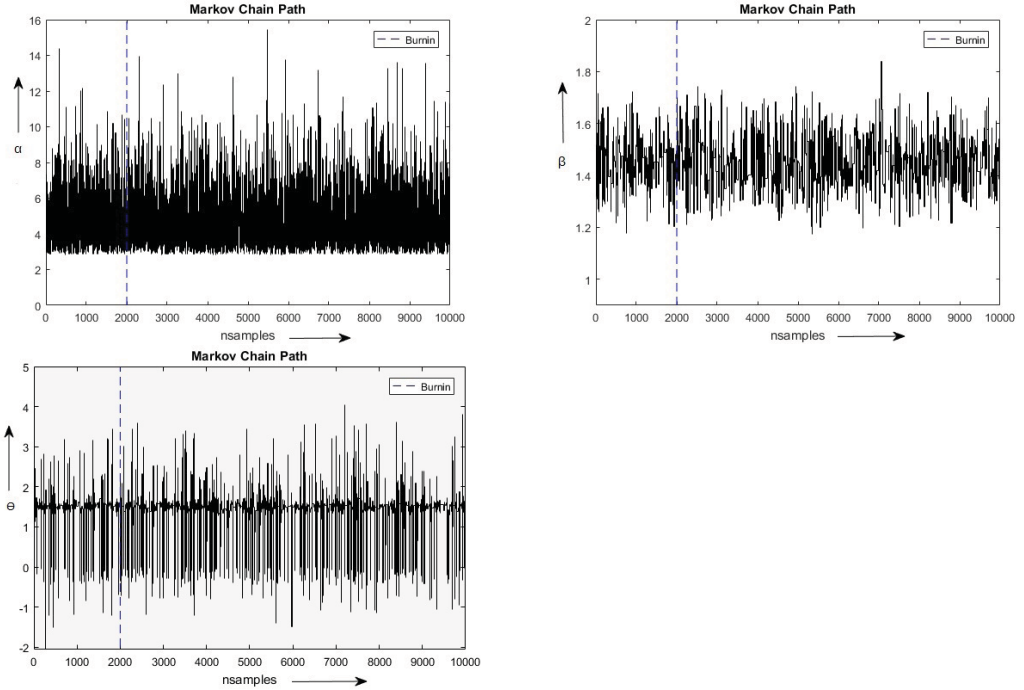


Figure 2: Trace plots for α, β and θ of Weibull Rayleigh Distribution

6 Monte Carlo Simulation Study

In this section, to compare the performance of various estimates using a simulation study. Here, hypothetical data is generated by using inverse CDF of the Weibull Rayleigh Distribution. Bayes estimates are evaluated using Gamma priors and the approximate bayesian estimate for α, β and θ are obtained by using MCMC method. The simulation study was carried out by using the algorithm provided by Balakrishnan and Sandhu(1995) to generate samples from Weibull Rayleigh distribution. Samples of size 10,20,40,60,80,100 were taken to generate data set from WR distribution with initial parameters, $\alpha = 1.3, \beta = 1.2$ and $\theta = 1.2$. In the simulation study, 1000 MCMC samples were generated from the posterior conditional densities of α, β and θ and was iterated 1000 times. The MLE and Bayesian estimates were computed on the basis of 1000 MCMC samples with the whole procedure was iterated 1000 times. The performance of the ML estimates and bayesian estimates under different loss function can be compared by computing mean square error given as

$$MSE = \frac{1}{1000} \sum_{i=1}^{1000} (\hat{\theta}^{(i)} - \theta_0)^2$$

where θ_0 are true values and the $\hat{\theta}^{(t)}$ are the t^{th} estimates of θ_0 evaluated at $\hat{\theta}$.

Table 4: MSE of MLE and Bayesian Estimates of unknown parameters under different loss function for $\omega = 0.6$

n	Parameters	MLE	BSEL	BLINEX	
				$q = -1$	$q = 1$
10	α	9.5001	4.4282	8.4672	1.9978
	β	0.2548	0.1884	0.2062	0.1661
	θ	1.3566	0.8744	1.1517	0.6563
20	α	4.4698	2.6903	3.8752	1.3350
	β	0.0955	0.0831	0.0860	0.0798
	θ	0.7045	0.6196	0.6341	0.6242
40	α	2.5533	1.8476	2.2678	1.2878
	β	0.0445	0.0413	0.0420	0.0404
	θ	0.3085	0.3077	0.3456	0.2896
60	α	1.1391	0.9648	1.0300	0.8704
	β	0.0268	0.0254	0.0257	0.0250
	θ	0.1802	0.1891	0.2304	0.1699
80	α	0.9570	0.8464	0.8805	0.7974
	β	0.0238	0.0228	0.0231	0.0226
	θ	0.1802	0.1729	0.2115	0.1549
100	α	0.8779	0.7912	0.8186	0.7458
	β	0.0192	0.0185	0.0187	0.0183
	θ	0.1420	0.1523	0.1924	0.1342

7 Conclusions

In this paper, we have performed bayesian estimation for the unknown parameters of Weibull Rayleigh distribution by using MCMC approach under BSEL and BLINEXL Function. The 95% Confidence and Credible Intervals corresponding to MLE and Bayesian estimates has also been obtained. We have generated different samples of size $n = 10(10)20(20)100$ form the Weibull Rayleigh distribution. From table 3, we can observe that mse values decrease with increase in sample size. From the result, it was observed that, MSE values from bayesian estimates are smaller than maximum likelihood estimates and hence bayesian estimation is more better than maximum likelihood estimation. Among different loss function, it can easily observe that bayes estimator under BLINEX loss function with shape parameter $\omega = 0.6$ and loss parameter $q = 1$ provide better estimates than other loss functions.

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