

New tail probability inequalities for widely orthant dependent random variables sequence, application to hazard estimator

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ABSTRACT

Tail probability inequalities have been an important tool in probability and statistics. Version of Bernstein type inequalities have proved for independent and for some dependence structure. We prove a new Tail probability inequality for the distributions of sums of widely orthant dependent (WOD, in short) random variables, and obtain of complete convergence for kernel estimators of density and hazard functions, under some suitable conditions.

Keywords: Orthant dependent, kernel estimators of density, hazard functions, complete convergence, exponential inequalities.

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1 Introduction

Hazard estimation is quite an important problem in several fields of applied statistics (medicine, econometric, seismic risk, reliability, etc.). Nonparametric estimation of the hazard function started with Watson and Leadbetter (1964a,b) who introduced the kernel estimator, and from that time on, a lot of papers on this topic have come out in the nonparametric literature. The estimation of the hazard function, in the nonparametric case, has been widely studied in the literature when the variables are of finite dimensions.

Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with an unknown marginal probability density function $f(x)$ and distribution function $F(x)$. Assume that $K(x)$ is a known kernel function, the kernel estimate of $f(x)$ and the empirical distribution function of $F(x)$ are given by

$$\tilde{f}_n(x) = \frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right), \quad F_n(x) = n^{-1} \sum_{i=1}^n \mathbb{I}(X_i < x), \quad (1.1)$$

where $\{h_n, n \geq 1\}$ is a sequence of positive bandwidths tending to zero as $n \rightarrow +\infty$. The function $\tilde{f}_n(\cdot)$ is the well known kernel estimator of $f(\cdot)$ defined by (Rosenblatt, 1956) and (Parzen, 1962). $\mathbb{I}(\cdot)$ is the indicator of the event specified in the parentheses. The hazard rate of function can be written as ratio between the $f(\cdot)$ and the survivor function $S(\cdot) = 1 - F(\cdot)$ as follows:

$$H(x) = f(x)/S(x),$$

and it can be estimated by

$$\tilde{H}_n(x) = \frac{\tilde{f}_n(x)}{1 - F_n(x)}. \quad (1.2)$$

In all these works, we establish the complete convergence of hazard function estimator based on widely orthant dependent (WOD in short) random variables by using the new exponential probability. It is well known that the probability limit theorem and its applications for independent random variables have been studied by many authors, while the assumption of independence is not reasonable in real practice. If the independent case is classical in the literature, the treatment of dependent random variables is more recent. Widely orthant dependence (WOD) is one of dependence structure. Now we recall the definition of WOD. The concept of widely orthant dependent was introduced by (Wang and Cheng, 2011) for risk model. They studied the basic renewal theorems for random walks with WOD dependent increments.

By definition, r.v.s $\{X_i, i \geq 1\}$, are said to be widely upper orthant dependent (WUOD) if for each $n \geq 1$, there exists a positive number $g_U(n)$ such that, for all $x_i \in (-\infty, +\infty), i = 1, \dots, n$

$$\mathbb{P}(X_1 > x_1, X_2 > x_2, \dots, X_n > x_n) \leq g_U(n) \prod_{i=1}^n \mathbb{P}(X_i > x_i); \quad (1.3)$$

they are said to be widely lower orthant dependent (WLOD) if for each $n \geq 1$, there exists some finite positive number $g_L(n)$ such that, for all $x_i \in (-\infty, +\infty), i = 1, \dots, n$,

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2, \dots, X_n \leq x_n) \leq g_L(n) \prod_{i=1}^n \mathbb{P}(X_i \leq x_i); \quad (1.4)$$

and they are said to be widely orthant dependent (WOD) if they are both WUOD and WLOD. WUOD, WLOD and WOD r.v.s. are called by a joint name widely dependent r.v.s. and $g_U(n), g_L(n), n \geq 1$ are called dominating coefficients. Clearly, we have $g_U(n) \geq 1, g_L(n) \geq 1, n \geq 2$, and $g_U(1) = g_L(1) = 1$.

The concept of complete convergence was introduced by (Hsu and Robbins, 1947) as follows: a sequence $X_n, n \geq 1$ of random variables converges completely to a constant ϑ if for all $\varepsilon > 0$,

$$\sum_{i=1}^n \mathbb{P}(|X_i - \vartheta| \geq \varepsilon) < +\infty.$$

By the Borel-Cantelli lemma, this implies that $X_n \rightarrow \vartheta$ a.s., and so complete convergence is a stronger concept than a.s. convergence.

In this paper, we attempt to establish a new probability inequality and to derive the complete convergence for the estimators of the probability density estimator (1.1) and hazard functions under strictly stationary WOD random variables. The results obtained in the paper improve and extend the corresponding ones of (Li and Yang, 2005) for NA samples. We will also study the complete consistency for the estimator (1.1) under some mild conditions. The main results

are presented in Section 2. Some lemmas are provided in Section 3. The proofs are given in Section 4. $\mathcal{C}(f)$ denotes all the continuity points of function f and $\mathcal{C}^2(f)$ stands for a point set in where the second-order derivative f'' exist and is bounded and continuous.

2 Some lemmas

To prove the main results of the paper, we need the following important lemmas. The first one is a basic property for WOD random variables, which was obtained by (Wang, Wang and Gao, 2013) .

Lemma 2.1.

- (i) Let $\{X_n, n \geq 1\}$ be WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$),
- if $\{f_n(\cdot), n \geq 1\}$ are nondecreasing, then $\{f_n(X_n), n \geq 1\}$ are still WLOD (WUOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$);
 - if $\{f_n(\cdot), n \geq 1\}$ are nonincreasing, then $\{f_n(X_n), n \geq 1\}$ are WUOD (WLOD) with dominating coefficients $g_L(n), n \geq 1$ ($g_U(n), n \geq 1$).
- (ii) If $\{X_n, n \geq 1\}$ are nonnegative and WUOD with dominating coefficients $g_U(n), n \geq 1$, then for each $n \geq 1$,

$$\mathbb{E} \prod_{i=1}^n X_i \leq g_U(n) \prod_{i=1}^n \mathbb{E} X_i.$$

In particular, if $\{X_n, n \geq 1\}$ are WUOD with dominating coefficients $g_U(n), n \geq 1$, then for each $n \geq 1$ and any $\lambda > 0$,

$$\mathbb{E} \exp\left\{\lambda \sum_{i=1}^n X_i\right\} \leq g_U(n) \prod_{i=1}^n \mathbb{E} \exp\{\lambda X_i\}.$$

By Lemma (2.1), we can get the following corollary immediately.

Corollary 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables.

- (i) If $\{f_n(\cdot), n \geq 1\}$ are all nondecreasing (or all nonincreasing), then $\{f_n(X_n), n \geq 1\}$ are still WOD.
- (ii) For each $n \geq 1$ and any $\lambda \in \mathbb{R}$,

$$\mathbb{E} \exp\left\{\lambda \sum_{i=1}^n X_i\right\} \leq g(n) \prod_{i=1}^n \mathbb{E} \exp\{\lambda X_i\}.$$

Proof. For $\lambda > 0$, it is easy to see that λX_i and $\lambda \sum_{j=i+1}^n X_j$ are WOD by the definition. Which

implies that $\exp(\lambda X_i)$ and $\exp(\lambda \sum_{j=i+1}^n X_j)$ are also WOD for $i = 1, 2, \dots, n-1$, by Lemma (2.1)

(i). It follows from corollary 2.2 we obtain

$$\begin{aligned}
 \mathbb{E}\left(\prod_{i=1}^n e^{\lambda X_i}\right) &= \mathbb{E}(\exp(\lambda X_1) \exp(\lambda \sum_{j=2}^n X_j)), \\
 &\leq g_1(n) \mathbb{E}[\exp(\lambda X_1)] \mathbb{E}[\exp(\lambda \sum_{j=2}^n X_j)], \\
 &= g_1(n) \mathbb{E}[\exp(\lambda X_1)] \mathbb{E}[\exp(\lambda X_2) \exp(\lambda \sum_{j=3}^n X_j)], \\
 &\leq g_1(n) g_2(n) \mathbb{E}[\exp(\lambda X_1)] \mathbb{E}[\exp(\lambda X_2)] \mathbb{E}[\exp(\lambda \sum_{j=3}^n X_j)], \\
 &\leq \prod_{i=1}^{n-1} g_i(n) \prod_{i=1}^n (\mathbb{E} e^{\lambda X_i}), \\
 &= g(n) \prod_{i=1}^n (\mathbb{E} e^{\lambda X_i}),
 \end{aligned}$$

where $g(n) = \prod_{i=1}^{n-1} g_i(n)$. The following lemma is very useful in the proof of Lemma (2.4). $\square$

Lemma 2.3. (Chen and Liu, 2017) For any $x \in \mathbb{R}$, we have

$$\exp(x) \leq 1 + x + |x|^{1+\alpha} \exp(2|x|), \quad 0 < \alpha \leq 1.$$

Lemma 2.4. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $\mathbb{E}X_n = 0$ for each $n \geq 1$. If there exists a sequence of positive numbers $\{a_n, n \geq 1\}$ such that $|X_i| \leq a_i$ for each $i \geq 1$, then for any $\lambda > 0$ and $0 < \alpha \leq 1$,

$$\mathbb{E} \exp \left\{ \lambda \sum_{i=1}^n X_i \right\} \leq g(n) \exp \left\{ \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|X_i|^{1+\alpha} \right\}. \quad (2.1)$$

Proof. By lemma 2.3, for all $x \in \mathbb{R}$ and $0 < \alpha \leq 1$, $\exp(x) \leq 1 + x + |x|^{1+\alpha} \exp(2|x|)$. Thus, by $\mathbb{E}(X_i) = 0$ and $|X_i| \leq a_i$ for each $i \geq 1$, we have

$$\begin{aligned}
 \mathbb{E} \exp(\lambda X_i) &\leq \mathbb{E} \{ 1 + \lambda X_i + |\lambda X_i|^{1+\alpha} \exp(2|\lambda X_i|) \} \\
 &= 1 + \lambda \mathbb{E}(X_i) + \lambda^{1+\alpha} \mathbb{E} \{ |X_i|^{1+\alpha} \exp(2|\lambda X_i|) \} \\
 &= 1 + \lambda^{1+\alpha} \mathbb{E} \{ |X_i|^{1+\alpha} \exp(2|\lambda X_i|) \} \\
 &\leq 1 + \lambda^{1+\alpha} \mathbb{E} \{ |X_i|^{1+\alpha} \exp(2\lambda a_i) \} \\
 &= 1 + \lambda^{1+\alpha} \exp(2\lambda a_i) \mathbb{E} \{ |X_i|^{1+\alpha} \} \\
 &\leq \exp \{ \lambda^{1+\alpha} \exp(2\lambda a_i) \mathbb{E} \{ |X_i|^{1+\alpha} \} \}
 \end{aligned} \quad (2.2)$$

(using $1 + y \leq \exp(y)$ for all $y \in \mathbb{R}$) for any $\lambda > 0$. By lemma (2.1) and (2.2), we have can see that

$$\mathbb{E} \exp \left\{ \lambda \sum_{i=1}^n X_i \right\} \leq g(n) \prod_{i=1}^n \mathbb{E} \exp \{ \lambda X_i \} \quad (2.3)$$

$$\leq g(n) \exp \left\{ \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|X_i|^{1+\alpha} \right\}. \quad (2.4)$$

The lemma is thus proved. $\square$

Lemma 2.5. Let $\{X_n, n \geq 1\}$ be a sequence of WOD random variables with $\mathbb{E}X_n = 0$ for each $n \geq 1$. If there exists a sequence of positive numbers $\{a_n, n \geq 1\}$ such that $|X_i| \leq a_i$ for each $i \geq 1$, then for any $\lambda > 0, 0 < \alpha \leq 1$ and $\epsilon > 0$

$$\mathbb{P}\left(\left|\sum_{i=1}^n X_i\right| \geq \epsilon\right) \leq 2g(n) \exp\left\{-\lambda\epsilon + \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|X_i|^{1+\alpha}\right\}. \quad (2.5)$$

Proof. By Markov's inequality and lemma (2.4), we can see that

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^n X_i \geq \epsilon\right) &\leq \exp(-\lambda\epsilon) \mathbb{E} \exp\left\{\lambda \sum_{i=1}^n X_i\right\} \\ &\leq \exp(-\lambda\epsilon) g(n) \prod_{i=1}^n \mathbb{E} \exp\{\lambda X_i\} \\ &\leq g(n) \exp\left\{-\lambda\epsilon + \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|X_i|^{1+\alpha}\right\}. \end{aligned} \quad (2.6)$$

The desired result follows by replacing X_i by $-X_i$ in (2.6). This completes the proof of the lemma. $\square$

Lemma 2.6. (Shanchao, 2003) Suppose that (H_1) holds, then for all $x \in C^2(f)$,

$$\lim_{h \rightarrow 0} \int_{\mathbb{R}} K(u) f(x - hu) du = f(x).$$

Lemma 2.7. Suppose that (H_1) holds, then for all $x \in C^2(f)$,

$$h_n^{-2} |\mathbb{E} \tilde{f}_n(x) - f(x)| \leq C < +\infty.$$

Proof. By Taylor's expansion and $(H_1) - (H_2)$, for any $0 < \theta < 1$ we can see that

$$\begin{aligned} \mathbb{E} \tilde{f}_n(x) &= \mathbb{E}\left(\frac{1}{nh_n} \sum_{i=1}^n K\left(\frac{x - X_i}{h_n}\right)\right) \\ &= \frac{1}{nh_n} \sum_{i=1}^n \mathbb{E} K\left(\frac{x - X_i}{h_n}\right) \\ &= \frac{1}{nh_n} \sum_{i=1}^n \int_{\mathbb{R}} K(u) f(x - h_n u) du \\ &= \frac{1}{nh_n} \sum_{i=1}^n \int_{\mathbb{R}} K(u) \left[f(x) - f'(x) h_n u + \frac{f''(x - \theta h_n u)}{2} (h_n u)^2\right] du \\ &= f(x) + \frac{h_n^2}{2} \int_{\mathbb{R}} K(u) f''(x - \theta h_n u) u^2 du \rightarrow f(x) + \frac{h_n^2}{2} f''(x) \int_{\mathbb{R}} K(u) u^2 du. \end{aligned} \quad (2.7)$$

By assumption (H_1) and (2.7), we get the result. $\square$

Lemma 2.8. Let $\{X_n; n \geq 1\}$ be a sequence of WOD random variables with unknown distribution function $F(x)$ and bounded probability density function $f(x)$. Let $F_n(x)$ be the empirical distribution function. If $\tilde{\xi}_n = \frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)}(ng(n))}{n^{1+\alpha}} \rightarrow 0$ for some $p > 1$ and $0 < \alpha \leq 1$, then

$$\sup_x |F_n(x) - F(x)| = O(\tilde{\xi}_n), \text{ completely.}$$

In particular, if $g(n) = O(n^\delta)$ for some $\delta \geq 0$, then

$$\sup_x |F_n(x) - F(x)| = O\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} n}{n^{1+\alpha}}\right), \text{ completely.}$$

Proof. Let $F(x_{ni}) = i/n$ for $n \geq 3$ and $1 \leq i \leq n-1$. By lemma 2 in (Shanchao, 2003) we have that

$$\sup_{x \in \mathbb{R}} |F_n(x) - F(x)| \leq \max_{1 \leq j \leq n-1} |F_n(x_{nj}) - F(x_{nj})| + 2/n. \quad (2.8)$$

Noting that $n\tilde{\xi}_n \rightarrow \infty$, then for any positive constant B_1 , we have that $2/n < B_1\tilde{\xi}_n/2$ for all n large enough. Then it follows from (2.8) that

$$\begin{aligned} \mathbb{P}(\sup_x |F_n(x) - F(x)| \geq B_1\tilde{\xi}_n) &\leq \mathbb{P}(\max_{1 \leq j \leq n-1} |F_n(x_{nj}) - F(x_{nj})| \geq B_1\tilde{\xi}_n/2) \\ &\leq \sum_{j=1}^{n-1} \mathbb{P}(|F_n(x_{nj}) - F(x_{nj})| \geq B_1\tilde{\xi}_n/2). \end{aligned} \quad (2.9)$$

Let $\tilde{Y}_i = I(X_i < x_{nj}) - \mathbb{E}I(X_i < x_{nj})$. By Lemma (2.1), $\{\tilde{Y}_i; i \geq 1\}$ is still a sequence of WOD random variables with $\mathbb{E}\tilde{Y}_i = 0$, $|\tilde{Y}_i| \leq 2$. Thus by choosing $\lambda = \left(\frac{B_1 b p}{2^{p(1+\alpha)+1}(1+\alpha)}\right)^{\frac{1}{p(1+\alpha)-1}} \left(\frac{\tilde{\xi}_n}{n^{p-1}}\right)^{\frac{1}{p(1+\alpha)-1}}$ in Lemma (2.5) we have that for all n large enough,

$$\begin{aligned} \mathbb{P}(|F_n(x_{nj}) - F(x_{nj})| \geq B_1\tilde{\xi}_n/2) &\leq \mathbb{P}\left(\left|\sum_{j=1}^n \tilde{Y}_i\right| \geq B_1\tilde{\xi}_n/2\right) \\ &\leq 2g(n) \exp\left\{-\lambda B_1\tilde{\xi}_n/2 + \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|\tilde{Y}_i|^{1+\alpha}\right\} \\ &\leq 2g(n) \exp\left\{-\lambda B_1\tilde{\xi}_n/2 + \lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} \mathbb{E}|\tilde{Y}_i|^{1+\alpha}\right\} \\ &\leq 2g(n) \exp\left\{-\lambda B_1\tilde{\xi}_n/2 + \lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} \right. \\ &\quad \left. 2^\alpha (\mathbb{E}(I(X_i < x_{nj}))^{1+\alpha}) + (\mathbb{E}I(X_i < x_{nj}))^{1+\alpha}\right\} \\ &\quad \text{(using the inequality } (a+b)^{\alpha+1} \leq \max(2^\alpha, 1)(a^{\alpha+1} + b^{\alpha+1})\text{)} \\ &\leq 2g(n) \exp\left\{-\lambda B_1\tilde{\xi}_n/2 + \lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} \right. \\ &\quad \left. 2^\alpha 2 \mathbb{E}(I(X_i < x_{nj}))^{1+\alpha}\right\} \\ &\quad \text{(using the Hölder's inequality } (\mathbb{E}(Z))^{1+\alpha} \leq \mathbb{E}(Z^{1+\alpha})\text{)} \\ &\leq 2g(n) \exp\left\{-\lambda B_1\tilde{\xi}_n/2 + \lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} 2^{\alpha+1}\right\}. \end{aligned}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0} \left\{ \frac{1}{pb} u^p + \frac{1}{q} b^{q/p} v^q \right\} \text{ for } u > 0, v > 0 \text{ and } 1/p + 1/q = 1.$$

This yields the inequality

$$\lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} 2^{\alpha+1} \leq \frac{1}{pb} \lambda^{p(1+\alpha)} n^p 2^{p(\alpha+1)} + \frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}. \quad (2.10)$$

We can thus conclude that for every $p > 1$ and $0 < \alpha \leq 1$, there for all $\lambda > 0$, such that

$$\begin{aligned} \mathbb{P}(|F_n(x_{nj}) - F(x_{nj})| \geq B_1 \tilde{\xi}_n / 2) &\leq 2g(n) \exp\{-\lambda B_1 \tilde{\xi}_n / 2 + \frac{1}{pb} \lambda^{p(1+\alpha)} n^p 2^{p(\alpha+1)}\} \\ &\times \exp\left\{-\frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}\right\} \\ &= 2g(n) \exp\left\{-\frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}\right\} \exp(\Phi(\lambda, n)). \end{aligned} \quad (2.11)$$

The equation $\frac{\partial \Phi(\lambda, n)}{\partial \lambda} = 0$ has the unique solution

$$\lambda = \left(\frac{B_1 b p}{2^{p(1+\alpha)+1}(1+\alpha)} \right)^{\frac{1}{p(1+\alpha)-1}} \left(\frac{\tilde{\xi}_n}{n^{p-1}} \right)^{\frac{1}{p(1+\alpha)-1}} \quad (2.12)$$

which minimizes $\Phi(\lambda, n)$. Then from (2.11), (2.12) and taking $2q\lambda \max_{1 \leq i \leq n} a_i \leq 1$ it follows that

$$\begin{aligned} \mathbb{P}(|F_n(x_{nj}) - F(x_{nj})| \geq B_1 \tilde{\xi}_n / 2) &\leq 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} g(n) \exp\left\{-\mu_1 n^{\frac{p\alpha}{p(1+\alpha)-1}} \tilde{\xi}_n^{\frac{p(1+\alpha)}{p(1+\alpha)-1}}\right\} \\ &\leq 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} g(n) \exp\{-\mu_1 C_0 \log(n g(n))\} \\ &= 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} g(n) \frac{1}{(n g(n))^{\mu_1 C_0}} \end{aligned} \quad (2.13)$$

Where $\mu_1 = (1 - \frac{1}{p(1+\alpha)})^{\frac{p(1+\alpha)}{2p(1+\alpha)-1}} (bp/(1+\alpha))^{\frac{1}{p(1+\alpha)-1}}$ Recall that $g(n) \geq 1$. Taking μ_1 sufficiently large such that $\mu_1 C_0 > 2$ by ((2.9)) and ((2.13)) we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}\left(\sup_{-\infty < x < +\infty} |F_n(x) - F(x)| \geq B_1 \tilde{\xi}_n\right) &\leq 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} \sum_{n=1}^{\infty} \sum_{i=1}^{n-1} g(n) \frac{1}{(n g(n))^{\mu_1 C_0}} \\ &= 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} \sum_{n=1}^{\infty} (n-1) g(n) \frac{1}{(n g(n))^{\mu_1 C_0}} \\ &\leq 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} \sum_{n=1}^{\infty} (n g(n)) \frac{1}{(n g(n))^{\mu_1 C_0}} \\ &= 2 \exp\left\{-\frac{1}{q} b^{q/p} e\right\} \sum_{n=1}^{\infty} \frac{1}{(n g(n))^{\mu_1 C_0 - 1}} \\ &< +\infty. \end{aligned}$$

This complete the proof of the lemma. □

3 Main Results and Proofs

In this section, we will present the complete convergence for kernel estimators of density and hazard functions. We adopt the following assumptions.

(H₁) $\int_{-\infty}^{+\infty} K(u)du = 1$, $\int_{-\infty}^{+\infty} uK(u)du = 0$, $\int_{-\infty}^{+\infty} u^2K(u)du < \infty$, $K(u) \in L_1$;

(H₂) The bandwidths h_n satisfy that $h_n \downarrow 0$ and $nh_n \rightarrow \infty$ as $n \rightarrow \infty$. Now we state our main results as follows.

Theorem 3.1. Suppose that (H₁) – (H₂) hold. Let $\{X_n; n \geq 1\}$ be a sequence of strictly stationary of WOD random variables with $g(n) = O(n^\delta)$ for some $\delta \geq 0$. Suppose that the kernel $K(\cdot)$ is a bounded monotone density function and the bandwidth $h_n = O(n^{-\frac{\alpha}{3(1+\alpha)} \log^{\frac{p(1+\alpha)-1}{3p(1+\alpha)}} n})$ for some $p > 1$ and $0 < \alpha \leq 1$. Then for any $x \in \mathcal{C}^2(f)$.

$$|\tilde{f}_n(x) - f(x)| = O\left(\frac{\log^{\frac{p(1+\alpha)-1}{p(1+\alpha)}} n}{n^{\frac{\alpha}{(1+\alpha)} h_n}}\right), \text{ completely.} \quad (3.1)$$

Proof. Set $Y_i = h_n^{-1} \left[K\left(\frac{x-X_i}{h_n}\right) - \mathbb{E}K\left(\frac{x-X_i}{h_n}\right) \right]$ for $1 \leq i \leq n$. Since $K(\cdot)$ is a bounded monotone density function, then $\{Y_i, i \geq 1\}$ is still a sequence of WOD random variables. Moreover, it follows from $0 < h_n \downarrow 0$ that there exists some positive constant $\tilde{C}$ such that $\max_{1 \leq i \leq n} |Y_i| \leq \tilde{C}h_n^{-1}$. By lemma (2.5) and elementary inequality $(a+b)^{1+\alpha} \leq \max(2^\alpha, 1)(a^{1+\alpha} + b^{1+\alpha})$ we have that

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}|Y_i|^{1+\alpha} &= \sum_{i=1}^n \mathbb{E} \left| h_n^{-1} \left[K\left(\frac{x-X_i}{h_n}\right) - \mathbb{E}K\left(\frac{x-X_i}{h_n}\right) \right] \right|^{1+\alpha} \\ &\leq \sum_{i=1}^n \frac{2^\alpha}{h_n^{1+\alpha}} \mathbb{E} \left[\left(K\left(\frac{x-X_i}{h_n}\right) \right)^{1+\alpha} + \left(\mathbb{E}K\left(\frac{x-X_i}{h_n}\right) \right)^{1+\alpha} \right] \\ &= \frac{2^\alpha}{h_n^{1+\alpha}} \sum_{i=1}^n \left[\mathbb{E} \left(K\left(\frac{x-X_i}{h_n}\right) \right)^{1+\alpha} + \left(\mathbb{E}K\left(\frac{x-X_i}{h_n}\right) \right)^{1+\alpha} \right] \\ &\leq \frac{2^{\alpha+1}}{h_n^{1+\alpha}} \sum_{i=1}^n \mathbb{E} \left(K\left(\frac{x-X_i}{h_n}\right) \right)^{1+\alpha} \quad (\text{by Hölder's inequality}) \\ &= \frac{2^{\alpha+1}}{h_n^{1+\alpha}} n \mathbb{E} \left(K\left(\frac{x-X_1}{h_n}\right) \right)^{1+\alpha} \\ &= \frac{2^{\alpha+1}}{h_n^{1+\alpha}} n \int_{\mathbb{R}} \left(K\left(\frac{x-u}{h_n}\right) \right)^{1+\alpha} f(u) du \\ &= \frac{2^{\alpha+1}}{h_n^{1+\alpha}} n \int_{\mathbb{R}} K(u) f(x-h_n u) du \\ &\leq \tilde{C} n^{\frac{2^{\alpha+1}}{1+\alpha}}. \end{aligned}$$

Set $\zeta_n = \frac{\log^{\frac{p(1+\alpha)-1}{p(1+\alpha)}} n}{n^{\frac{\alpha}{(1+\alpha)} h_n}}$. Applying lemma (2.5) with $\lambda = \left(\frac{A_1 n \zeta_n h_n^{p(1+\alpha)}}{C_2 n^p p(1+\alpha)} \right)^{\frac{1}{p(1+\alpha)-1}}$, where A_1 is some positive constant, for all n large enough, then we get that

$$\begin{aligned}
 \mathbb{P}(|\tilde{f}_n(x) - \mathbb{E}\tilde{f}_n(x)| \geq A_1\zeta_n) &\leq \mathbb{P}\left(\left|\sum_{j=1}^n Y_j\right| \geq A_1n\zeta_n\right) \\
 &\leq 2g(n) \exp\left\{-\lambda A_1n\zeta_n + \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|Y_i|^{1+\alpha}\right\} \\
 &\leq 2g(n) \exp\left\{-\lambda A_1n\zeta_n + \lambda^{1+\alpha} e^{2\lambda \max_{1 \leq i \leq n} a_i} \mathbb{E}|Y_i|^{1+\alpha}\right\} \\
 &\leq 2g(n) \exp\{-\lambda A_1n\zeta_n + \lambda^{1+\alpha} n^{2^{1+\alpha}} \tilde{C} h_n^{-(1+\alpha)} e^{2\lambda \max_{1 \leq i \leq n} a_i}\}.
 \end{aligned}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0} \left\{ \frac{1}{pb} u^p + \frac{1}{q} b^{q/p} v^q \right\} \text{ for } u > 0, v > 0 \text{ and } 1/p + 1/q = 1.$$

This yields the inequality

$$\lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} 2^{\alpha+1} \tilde{C} h_n^{-(1+\alpha)} \leq \frac{1}{pb} \lambda^{p(1+\alpha)} n^p 2^{p(\alpha+1)} \tilde{C}^p h_n^{-p(1+\alpha)} + \frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}. \quad (3.2)$$

We can thus conclude that for every $p > 1$ and $0 < \alpha \leq 1$, there for all $\lambda > 0$, such that

$$\begin{aligned}
 \mathbb{P}(|\tilde{f}_n(x) - \mathbb{E}\tilde{f}_n(x)| \geq A_1\zeta_n) &\leq 2g(n) \exp\left\{-\lambda A_1n\zeta_n + \frac{1}{pb} \lambda^{p(1+\alpha)} n^p 2^{p(1+\alpha)} \tilde{C}^p h_n^{-p(1+\alpha)}\right\} \\
 &\times \exp\left\{\frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}\right\} \\
 &= 2g(n) \exp\left\{\frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}\right\} \exp(\Psi(\lambda, n)).
 \end{aligned} \quad (3.3)$$

The equation $\frac{\partial \Psi(\lambda, n)}{\partial \lambda} = 0$ has the unique solution

$$\lambda = \left(\frac{A_1\zeta_n b p}{2^{p(1+\alpha)} (1+\alpha) n^{p-1} \tilde{C}^p h_n^{-p(1+\alpha)}} \right)^{\frac{1}{p(1+\alpha)-1}} \quad (3.4)$$

which minimizes $\Phi(\lambda, n)$. Then from (3.3), (3.4) and noting $2q\lambda \max_{1 \leq i \leq n} a_i \leq 1$, we obtain upper bound for the tail probability as

$$\begin{aligned}
 \mathbb{P}\left(\left|\sum_{j=1}^n Y_j\right| \geq A_1n\zeta_n\right) &\leq 2g(n) \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{A_1\zeta_n b p}{2^{p(1+\alpha)} (1+\alpha) n^{p-1} \tilde{C}^p h_n^{-p(1+\alpha)}}\right)^{\frac{1}{p(1+\alpha)-1}}\right. \\
 &\quad \left.A_1n\zeta_n \left(1 - \frac{1}{p(1+\alpha)}\right)\right\} \\
 &\leq 2g(n) \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\tilde{M} n^{\frac{p\alpha}{p(1+\alpha)-1}} \zeta_n^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} h_n^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right)\right\} \\
 &\leq 2M_1 \exp\left\{\frac{1}{q} b^{q/p} e\right\} n^{\delta} \exp\left\{-\tilde{M} \left(1 - \frac{1}{p(1+\alpha)}\right) \log n\right\} \\
 &= 2M_1 \exp\left\{\frac{1}{q} b^{q/p} e\right\} n^{\delta - \tilde{M} \left(1 - \frac{1}{p(1+\alpha)}\right)}.
 \end{aligned}$$

Taking $\widetilde{M} = (1 - \frac{1}{p(1+\alpha)})A_1 \frac{p(1+\alpha)}{p(1+\alpha)-1} \frac{bp}{C_2 p^{p(1+\alpha)}}$ large enough such that $\delta - \widetilde{M}(1 - \frac{1}{p(1+\alpha)}) < -1$, then we have that

$$\sum_{j=1}^n \mathbb{P}(|\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x)| \geq A_1 \zeta_n) < +\infty, \quad (3.5)$$

that is

$$|\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x)| = O\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} n}{n^{\frac{\alpha}{(1+\alpha)}} h_n}\right), \text{ completely.} \quad (3.6)$$

By lemma (2.7) that

$$h_n^{-2} |\mathbb{E}\widetilde{f}_n(x) - f(x)| \leq C < +\infty.$$

Which implies that

$$|\mathbb{E}\widetilde{f}_n(x) - f(x)| = O\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} n}{n^{\frac{\alpha}{(1+\alpha)}} h_n}\right). \quad (3.7)$$

Note that

$$\begin{aligned} |\widetilde{f}_n(x) - f(x)| &= |\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x) + \mathbb{E}\widetilde{f}_n(x) - f(x)| \\ &\leq |\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x)| + |\mathbb{E}\widetilde{f}_n(x) - f(x)|. \end{aligned}$$

Therefore, the desired result (3.1) follows immediately by (3.6)-(3.7). The proof is completed. $\square$

Theorem 3.2. Suppose that $(H_1) - (H_2)$ hold. $\beta_n = \left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} (ng(n))}{n^{\frac{\alpha}{(1+\alpha)}} h_n}\right) \rightarrow 0$. Let $\{X_n; n \geq 1\}$ be a sequence of strictly stationary of WOD random variables. Suppose that the kernel $K(\cdot)$ is a bounded monotone density function and the bandwidth satisfies that $h_n = O(n^{-\frac{\alpha}{3(1+\alpha)}} \log^{\frac{p(1+\alpha)-1}{3p(1+\alpha)}} n)$ for some $p > 1$ and $0 < \alpha \leq 1$. Then for any $x \in \mathcal{C}^2(f)$.

$$|\widetilde{f}_n(x) - f(x)| = O(\beta_n), \text{ completely.} \quad (3.8)$$

Proof. In view of the proof of Theorem (3.1), we only need to show that

$$|\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x)| = O\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} (ng(n))}{n^{\frac{\alpha}{(1+\alpha)}} h_n}\right), \text{ completely.} \quad (3.9)$$

and

$$|\mathbb{E}\widetilde{f}_n(x) - f(x)| = O\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} (ng(n))}{n^{\frac{\alpha}{(1+\alpha)}} h_n}\right). \quad (3.10)$$

Noting that $g(n) \geq 1$ then (3.10) follows from (3.10) immediately and thus we only need to prove (3.9). Similar to the proof of (3.9), set $\beta_n = \left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} (ng(n))}{n^{\frac{\alpha}{(1+\alpha)}} h_n}\right)$. Applying Lemma (2.5)

with $\lambda = \left(\frac{A_2 n \beta_n h_n^{p(1+\alpha)}}{C_2 n^p p(1+\alpha)}\right)^{\frac{1}{p(1+\alpha)-1}}$ to obtain that for all n large enough,

$$\begin{aligned}
 \mathbb{P}(|\tilde{f}_n(x) - \mathbb{E}\tilde{f}_n(x)| \geq A_2\beta_n) &\leq \mathbb{P}\left(\left|\sum_{j=1}^n Y_i\right| \geq A_2n\beta_n\right) \\
 &\leq 2g(n) \exp\left\{-\lambda A_2n\beta_n + \lambda^{1+\alpha} \sum_{i=1}^n e^{2\lambda a_i} \mathbb{E}|Y_i|^{1+\alpha}\right\} \\
 &\leq 2g(n) \exp\left\{-\lambda A_2n\beta_n + \lambda^{1+\alpha} e^{2\lambda \max_{1 \leq i \leq n} a_i} \mathbb{E}|Y_i|^{1+\alpha}\right\} \\
 &\leq 2g(n) \exp\{-\lambda A_2n\beta_n + \lambda^{1+\alpha} n 2^{1+\alpha} \tilde{C} h_n^{-(1+\alpha)} e^{2\lambda \max_{1 \leq i \leq n} a_i}\}.
 \end{aligned}$$

Let $p > 1$. It is well known that

$$uv = \inf_{b>0} \left\{ \frac{1}{pb} u^p + \frac{1}{q} b^{q/p} v^q \right\} \text{ for } u > 0, v > 0 \text{ and } 1/p + 1/q = 1.$$

This yields the inequality

$$\lambda^{1+\alpha} n e^{2\lambda \max_{1 \leq i \leq n} a_i} 2^{\alpha+1} \tilde{C} h_n^{-(1+\alpha)} \leq \frac{1}{pb} \lambda^{p(1+\alpha)} n^p 2^{p(\alpha+1)} \tilde{C}^p h_n^{-p(1+\alpha)} + \frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}. \quad (3.11)$$

We can thus conclude that for every $p > 1$ and $0 < \alpha \leq 1$, there for all $\lambda > 0$, such that

$$\begin{aligned}
 \mathbb{P}(|\tilde{f}_n(x) - \mathbb{E}\tilde{f}_n(x)| \geq A_2\beta_n) &\leq 2g(n) \exp\{-\lambda A_2n\beta_n + \frac{1}{pb} \lambda^{p(1+\alpha)} n^p 2^{p(1+\alpha)} \tilde{C}^p h_n^{-p(1+\alpha)}\} \\
 &\times \exp\left\{-\frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}\right\} \\
 &= 2g(n) \exp\left\{-\frac{1}{q} b^{q/p} e^{2q\lambda \max_{1 \leq i \leq n} a_i}\right\} \exp(\tilde{\Psi}(\lambda, n)).
 \end{aligned} \quad (3.12)$$

The equation $\frac{\partial \tilde{\Psi}(\lambda, n)}{\partial \lambda} = 0$ has the unique solution

$$\lambda = \left(\frac{A_2\beta_n b p}{2^{p(1+\alpha)} (1+\alpha) n^{p-1} \tilde{C}^p h_n^{-p(1+\alpha)}} \right)^{\frac{1}{p(1+\alpha)-1}} \quad (3.13)$$

which minimizes $\tilde{\Psi}(\lambda, n)$. Then from (3.12), (3.13) and noting $2q\lambda \max_{1 \leq i \leq n} a_i \leq 1$, we obtain upper bound for the tail probability as

$$\begin{aligned}
 \mathbb{P}\left(\left|\sum_{j=1}^n Y_i\right| \geq A_2n\beta_n\right) &\leq 2g(n) \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\left(\frac{A_2\beta_n b p}{2^{p(1+\alpha)} (1+\alpha) n^{p-1} \tilde{C}^p h_n^{-p(1+\alpha)}}\right)^{\frac{1}{p(1+\alpha)-1}}\right. \\
 &\quad \left.A_2n\beta_n \left(1 - \frac{1}{p(1+\alpha)}\right)\right\} \\
 &\leq 2g(n) \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\tilde{M}_1 n^{\frac{p\alpha}{p(1+\alpha)-1}} \beta_n^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} h_n^{\frac{p(1+\alpha)}{p(1+\alpha)-1}} \left(1 - \frac{1}{p(1+\alpha)}\right)\right\} \\
 &\leq \frac{2}{n} (ng(n)) \exp\left\{\frac{1}{q} b^{q/p} e\right\} \exp\left\{-\tilde{M}_1 \left(1 - \frac{1}{p(1+\alpha)}\right) \log(ng(n))\right\} \\
 &= 2(ng(n)) \exp\left\{\frac{1}{q} b^{q/p} e\right\} (ng(n))^{-\tilde{M}_1 (1 - \frac{1}{p(1+\alpha)})} \\
 &= 2 \exp\left\{\frac{1}{q} b^{q/p} e\right\} (ng(n))^{1 - \tilde{M}_1 (1 - \frac{1}{p(1+\alpha)})}.
 \end{aligned}$$

Taking $\widetilde{M}_1 = (1 - \frac{1}{p(1+\alpha)})A_2 \frac{p(1+\alpha)}{p(1+\alpha)-1} \frac{bp}{C_{p2p(1+\alpha)}}$ large enough such that $1 - \widetilde{M}(1 - \frac{1}{p(1+\alpha)}) < -1$, then we have that

$$\sum_{j=1}^n \mathbb{P}(|\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x)| \geq A_2/\beta_n) < +\infty, \quad (3.14)$$

which is equivalent to (3.9). The proof is completed. $\square$

Furthermore, by relaxing the restriction on the bandwidth h_n , we have the following result.

Theorem 3.3. Suppose that $(H_1) - (H_2)$ hold. Let $\{X_n; n \geq 1\}$ be a sequence of strictly stationary of WOD random variables. Suppose that the kernel $K(\cdot)$ is a bounded monotone density function and $\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)}(ng(n))}{n^{\frac{\alpha}{(1+\alpha)}h_n}}\right) \rightarrow 0$. Then for any $x \in \mathcal{C}^2(f)$,

$$\widetilde{f}_n(x) - f(x) \rightarrow 0, \text{ completely.} \quad (3.15)$$

Proof. In view of the proof theorem (3.2), by (3.9) and $\left(\frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)}(ng(n))}{n^{\frac{\alpha}{(1+\alpha)}h_n}}\right) \rightarrow 0$ we have that

$$\widetilde{f}_n(x) - \mathbb{E}\widetilde{f}_n(x) \rightarrow 0, \text{ completely.}$$

Therefore, we only need to show that

$$|\mathbb{E}\widetilde{f}_n(x) - f(x)| \rightarrow 0 \quad (3.16)$$

without the condition $h_n = O(n^{-\frac{\alpha}{3(1+\alpha)}} \log^{\frac{p(1+\alpha)-1}{3p(1+\alpha)}} n)$ in theorem(3.2). Actually, by lemma(2.5) we have that

$$\lim_{n \rightarrow +\infty} |\mathbb{E}\widetilde{f}_n(x) - f(x)| \leq C \lim_{n \rightarrow +\infty} h_n^2 = 0.$$

Consequently, (3.16) is proved and thus the proof of the theorem is completed. $\square$

Since the proofs of Theorems (3.4)-(3.6) are similar, we only present the proof of Theorem 2.4 as follows.

As an application of the results above, we obtain the complete convergence and the complete convergence for the hazard function estimator $\widetilde{H}_n(x)$ as follows.

Theorem 3.4. Suppose that $(H_1) - (H_2)$ hold. Let $\{X_n; n \geq 1\}$ be a sequence of strictly stationary of WOD random variables with $g(n) = O(n^\delta)$ for some $\delta \geq 0$. Suppose that the kernel $K(\cdot)$ is a bounded monotone density function and the bandwidth satisfies that $h_n = O(n^{-\frac{\alpha}{3(1+\alpha)}} \log^{\frac{p(1+\alpha)-1}{3p(1+\alpha)}} n)$ for some $p > 1$ and $0 < \alpha \leq 1$. If there exists a point x_0 such that $F(x_0) < 1$, then for any $x \in \mathcal{C}^2(f)$ and $x \leq x_0$,

$$|\widetilde{H}_n(x) - H(x)| = O(n^{-\frac{\alpha}{(1+\alpha)}} h_n^{-1} \log^{\frac{p(1+\alpha)-1}{p(1+\alpha)}} n), \text{ completely.} \quad (3.17)$$

Proof. The proof of this theorem is based on the following decomposition:

$$\begin{aligned}
 |\tilde{H}_n(x) - H(x)| &\leq \left| \frac{\tilde{f}_n(x)}{1 - F_n(x)} - \frac{f(x)}{1 - F(x)} \right| \\
 &\leq \left| \frac{(1 - F_n(x))(\tilde{f}_n(x) - f(x)) + (F_n(x) - F(x))f(x)}{(1 - F_n(x))(1 - F(x))} \right| \\
 &\leq \frac{1}{1 - F_n(x)} |\tilde{f}_n(x) - f(x)| + \frac{f(x)}{(1 - F_n(x))(1 - F(x))} |F_n(x) - F(x)|
 \end{aligned} \tag{3.18}$$

From $0 \leq F(x) \leq F(x_0) < 1$ for all $x \leq x_0$, $\sup_x f(x) \leq M < +\infty$, applying theorem (3.1) and taking $\tilde{\xi}_n = \frac{\log \frac{p(1+\alpha)-1}{p(1+\alpha)} n}{n^{\frac{\alpha}{1+\alpha}}}$ in lemma (2.8), we can see that

$$|\tilde{f}_n(x) - f(x)| = O(n^{-\frac{\alpha}{1+\alpha}} h_n^{-1} \log^{\frac{p(1+\alpha)-1}{p(1+\alpha)}} n), \text{ completely.} \tag{3.19}$$

and

$$\sup_{x \leq x_0} |F_n(x) - F(x)| = O(n^{-\frac{\alpha}{1+\alpha}} \log^{\frac{p(1+\alpha)-1}{p(1+\alpha)}} n), \text{ completely.} \tag{3.20}$$

On the other hand, from (3.20) one has that $x \leq x_0$ and all n large enough,

$$1 - F_n(x) \geq 1 - F(x) \geq \frac{1 - F(x_0)}{2} > 0. \tag{3.21}$$

consequently, the desired result (3.17) follows from (3.18) – (3.21). The proof is completed. Since the proof of theorems (3.4)-(3.6) are similar, we only present the proof of theorem (3.4) as follows. $\square$

Theorem 3.5. Suppose that $(H_1) - (H_2)$ hold. $\beta_n = \left(\log \frac{p(1+\alpha)-1}{p(1+\alpha)} \frac{(ng(n))}{n^{\frac{\alpha}{1+\alpha}} h_n} \right) \rightarrow 0$. Let $\{X_n; n \geq 1\}$ be a sequence of strictly stationary of WOD random variables. Suppose that the kernel $K(\cdot)$ is a bounded monotone density function and the bandwidth satisfies that $h_n = O(n^{-\frac{\alpha}{3(1+\alpha)} \log^{\frac{p(1+\alpha)-1}{3p(1+\alpha)}} n})$ for some $p > 1$ and $0 < \alpha \leq 1$. Then for any $x \in \mathcal{C}^2(f)$ and $x \leq x_0$,

$$|\tilde{H}_n(x) - H(x)| = O(\beta_n), \text{ completely.} \tag{3.22}$$

Theorem 3.6. Suppose that $(H_1) - (H_2)$ hold. Let $\{X_n; n \geq 1\}$ be a sequence of strictly stationary of WOD random variables. Suppose that the kernel $K(\cdot)$ is a bounded monotone density function and $\log \frac{p(1+\alpha)-1}{p(1+\alpha)} \frac{(ng(n))}{n^{\frac{\alpha}{1+\alpha}} h_n} \rightarrow 0$. If there exists a point x_0 such that $F(x_0) < 1$, then for any $x \in \mathcal{C}^2(f)$ and $x \leq x_0$,

$$\tilde{H}_n(x) - H(x) \rightarrow 0, \text{ completely.} \tag{3.23}$$

Conclusion

Our work consists in establishing some new exponential inequalities for the distribution of sums of WOD random variables. Using these inequalities, we proved the complete convergence for kernel estimators of density and hazard functions, under some suitable conditions.

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